Stable rule or brittle power? On exponential rule lengths and Roman Emperors

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Abstract

In this paper we demonstrate that lengths of rule of Roman Emperors were exponentially distributed, implying that their reigns ceased unexpectedly, without accumulation of prior tensions, political or economic damages, etc. A distribution-free goodness of fit test of exponentiality is used, based on a transformed empirical process. That test is complemented by further techniques which make use of age at ascent to rule.

Keywords: Distribution free statistics; Durations of rule; Remaining life distributions; Goodness of fit statistics; Local alternatives; Transformed empirical processes; Chinese Emperors; European monarchies

1 Introduction and main observation

This paper is not so much on a novel statistical technique as on an unusual phenomenon that was discovered using a combination of statistical approaches.

A fundamental reason for fitting a statistical model to observations is to gain insight into the nature of the random phenomena behind the data, and so help explain the data generating process. If that model has some clearly understood random mechanism behind it, the insight gained is that much deeper. The exponential distribution, with parameter $\lambda > 0$,

$$F(x, \lambda) = 1 - e^{-\lambda x}, \quad x \geq 0, \tag{1}$$

has a well known and characteristic lack of memory property. For example, if the occurrences of certain events (e.g., emissions of radioactive particles, traffic accidents, occurrence of abnormally high prices on stable markets, etc.) take place at random times and if the random elapsed times between them have an exponential distribution, then these events occur purely at random, in an unexpected and unpredictable way.

It seems clear that the life length of human beings can not be exponentially distributed; humans suffer from illnesses and stresses, accumulate damages and, in a word, are aging. Apart from fatal accidents, humans typically die as a result of this aging process.

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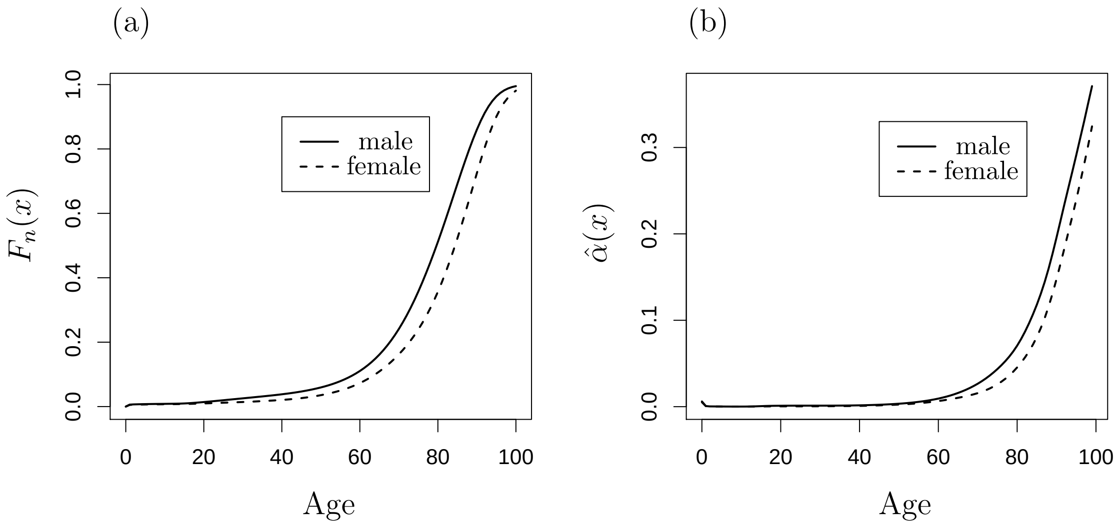

Figure 1: New Zealand males and females, 2000-2002: (a) empirical distribution functions of life length, (b) empirical rates of mortality. Data from Statistics New Zealand (2004).

As an illustration of the effects of aging consider Fig. 1 (a), which shows the empirical distribution functions, $F_n(x)$, of the life length of New Zealand males and females (Statistics New Zealand, 2004). There is no resemblance to the form of an exponential distribution, which is a concave function for any $\lambda > 0$. Fig. 1 (b) shows an empirical estimate of the corresponding failure rate, hazard rate or force of mortality, $\alpha(x) = f(x)/[1 - F(x)]$ with $f(x) = dF(x)/dx$. Again there is no resemblance to the constant failure rate $\alpha(x) = \lambda$, which characterizes an exponential distribution.

It seemed to us that the situation with historical rulers, kings, governors and so on would be similar: they would have stopped ruling, reigning and governing as a consequence of accumulated controversies and tensions, plus damages of economic, social, political or personal nature. Contrary to this, we found that data on the lengths of rule of Roman Emperors showed agreement with the exponential distribution, which seems remarkably good for such complex and long spanned historical data. Hence we are led to the surprising conclusion that Roman Emperors had stopped ruling (and in many cases, died) 'purely at random', in unexpected and unpredictable ways, and not as a consequence of accumulated controversies or tensions.

We initially considered the era of the "decline and fall" (Gibbon, 1988), taken conditionally from Nerva (reign: 96-98) to Theodosius I (reign: 379-395), the last emperor of both the Eastern and Western Roman Empire. Our first impression was that exponentiality might be a statistical manifestation of the complex phenomena labelled 'decline and fall'. We suggest such an impression is false though, since agreement with exponentiality was also found in data extending back to the first Roman Emperor, Augustus (reign: 27BC-14AD). Further, the pattern of exponentiality can be found in a much wider range of historic times than those which can be called 'decline and fall' periods.

Indeed, in search of an alternative example of non-exponential reign lengths, we considered data for Chinese Emperors (from 770 BC to 1644 AD) throughout a colossal span of their enormous history. In contrast to our prior expectations the data again showed surprisingly good agreement with the pattern of the exponential distribution. However in European monarchies of recent times, starting in the 15th century for example, it is easy to see examples of non-exponential rule lengths.

In section 3 we illustrate how good the agreement can be between data and the exponential distribution, where we present some goodness of fit results using a test described in section 2. It is possible, especially in monarchies of later times, that we observe only a similarity in the pattern rather than genuine exponentiality of the distribution. In section 4 we consider two competing explanations for the observed exponential pattern and in section 5 we demonstrate both possibilities in historical data. For Roman Emperors though, we believe we indeed have exponentiality of their rule lengths, with all the interpretation that invokes.

2 Testing exponentiality

To test the hypothesis that a sample $T_1, \dots, T_n$ of n independent random variables follows an exponential distribution (1) with some a priori unspecified $\lambda > 0$, we used the approach suggested in Khmaladze (1981). The application of this approach to testing exponentiality was recently considered in detail in Haywood and Khmaladze (2005). Briefly, one could consider an empirical distribution function $F_n(t)$ of the sample and compare it to $F(t, \hat{\lambda})$, with λ estimated by $\hat{\lambda}$ from the same data. However, there is some advantage in considering not the difference $F_n(t) - F(t, \hat{\lambda})$ but instead $F_n(t) - K(t, F_n)$, where $K(t, F_n)$ is called the 'compensator' in the martingale theory of point processes (e.g., Liptser and Shiryaev, 2001, vol. II). $K(t, F_n)$ can be thought of as a somewhat more flexible modification of $F(t, \hat{\lambda})$; for the family of exponential distributions its form is particularly simple:

$$\begin{aligned} K(t, F_n) &= \hat{\lambda} \int_0^\infty \left(2 + \frac{\hat{\lambda}}{2} \min(t, \tau) - \hat{\lambda}\tau\right) \min(t, \tau) F_n(d\tau) \\ &= \frac{\hat{\lambda}}{n} \sum_{i=1}^n \left[2 \min(t, T_i) + \frac{\hat{\lambda}}{2} \min^2(t, T_i) - \hat{\lambda} T_i \min(t, T_i)\right] \end{aligned} \tag{2}$$

or

$$K(t, F_n) = \frac{\hat{\lambda}}{n} \sum_{i:T_i \le t} \left(2T_i - \frac{\hat{\lambda}}{2} T_i^2\right) + \hat{\lambda} \left(2 + \frac{\hat{\lambda}}{2} t\right) t[1 - F_n(t)] - t\frac{\hat{\lambda}^2}{n} \sum_{i:T_i > t} T_i \,.$$

To see that $K(t, F_n)$ is really a 'modification' of $F(t, \hat{\lambda})$, one can check that $K[t, F(t, \lambda)] = F(t, \lambda)$.

If the hypothesis of exponentiality is true, the version of the empirical process $w_n(s), s \in [0, 1]$, defined as

$$w_n(s) = \sqrt{n}[F_n(t) - K(t, F_n)], \quad s = F(t, \hat{\lambda}),$$

converges in distribution, as $n \to \infty$, to standard Brownian motion $w(s), s \in [0, 1]$. So for the distribution of the classical Kolmogorov-Smirnov statistic,

$$D_n = \sqrt{n} \sup_{t \ge 0} |F_n(t) - K(t, F_n)| = \sup_{0 \le s \le 1} |w_n(s)|,$$

we obtain, for large n and $z \ge 0$, the approximation

$$\mathbb{P}(D_n \le z) \approx \mathbb{P}(\sup_{0 \le s \le 1} |w(s)| \le z) = G(z).$$

The explicit expression for $G(z)$ is

$$G(z) = \frac{4}{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{2n + 1} \exp\left\{\frac{-\pi^2(2n + 1)^2}{8z^2}\right\},$$

which can be found, e.g., in Shiryaev (1999) and Borodin and Salminen (2002). Since w is a standard process the distribution $G(z)$ is also standard and depends neither on the form of $F(t, \lambda)$ nor on the value of λ. Hence we have obtained what is called an asymptotically distribution-free testing procedure.

Convergence of w_n to w is not slow and $G(z)$ can be used as an approximation of $\mathbb{P}(D_n \leq z)$ for relatively small n with sufficient confidence (see Haywood and Khmaladze, 2005, section 4). In particular, $G(1.960) = 0.90$, $G(2.241) = 0.95$ and $G(2.807) = 0.99$, while extensive tables of $G(z)$ can be found at `http://www.mcs.vuw.ac.nz/~ray/Brownian`.

3 Goodness of fit results

We examine three versions of the chronological tables of Roman Emperors. The first is the chronology given in Kienast (1990). The next was kindly made available to us by Prof. T. Parkin (personal communication, but see, e.g., Parkin, 1992). Parkin's chronology also contains birth dates and hence the ages at ascent, which we use in the analysis below. Finally, we use the chronology given in the classical work by Edward Gibbon (1988). Although these three chronologies are slightly different in certain details, the main conclusion about the exponential pattern of the durations of rule lengths remains, irrespective of the chronology used. We note that in each chronology there are instances of the simultaneous or parallel rule of two or more Emperors; this situation was not uncommon in the vast Roman Empire. All three chronologies are presented in the Appendix.

Chronology of Kienast (referred to as "K"). In the list of emperors from Augustus to Theodossius there are 64 names, with $n = 53$ in our period of interest starting at Nerva ('decline and fall'). The chronology shows the date of ascent and abdication (or death) in many cases. In some cases, however, no specific day or month is suggested. In these cases we selected mid-points, as described below for Parkin's chronology.

Chronology of Parkin (referred to as "P"). In the list of emperors from Augustus to Theodossius there are 70 names, with $n = 59$ starting at Nerva. In the majority of cases it agrees with the chronology "K", but there are a few differences. When several possible days or even several possible years are suggested, if the dates were not too far apart we selected mid-points or even 'average dates'. For example, dates "June 251 – August 253" for Trebonianus Gallus we interpreted as "15 June 251 − 15 August 253" when calculating the duration of reign in days. Relative to the total of 729 days, the possible error does not look large. Similarly, "early 244 – September or October 249" as reign dates for Philippus we interpreted as "30 January 244 – 30 September 249", which again does not involve a big error relative to the total of 2070 days, etc. One special case is Quintillus, who ascended and died in the same month (September 270), but no day-of-month is given for either event. Researchers 'agree' that the reign lasted at least 17 days (but some suggest as much as 77 days). So for Quintillus we specified a reign of 20 days. As historical statements concerning individual rulers these mid-points or averages would be impractical. Yet when considering the whole collection of rule lengths as a 'statistical ensemble', replacement of two or more uncertain points by a mid-point will deform the pattern very little.

Chronology of Gibbon (referred to as "G"). In the list of emperors from Augustus to Theodossius there are 63 names, with $n = 52$ starting at Nerva. This chronology specifies years of ascent and abdication, and we calculated the length of reign in years in a quite straightforward way, as a difference. Some years in this chronology are different from the other two and even the list of emperors is different. For example, "G" does not include Clodius Albinus, Pescennius Niger or Quintillus. There is much debate about who should

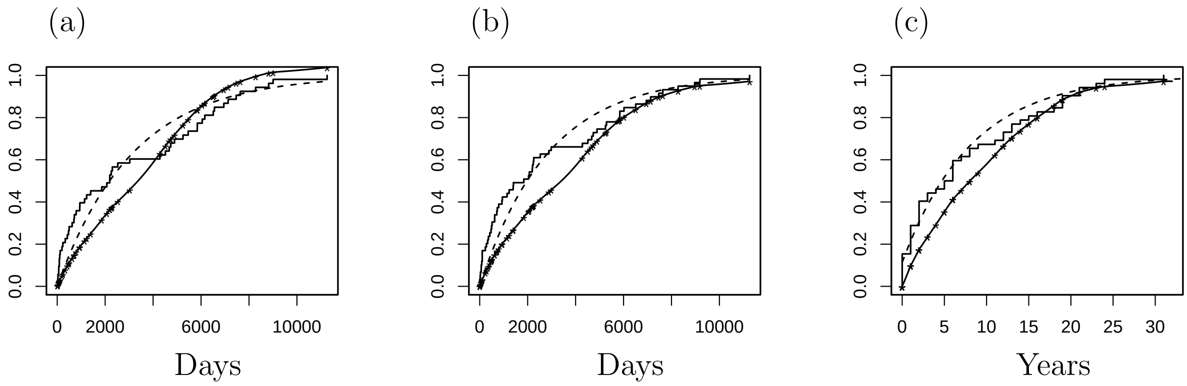

Figure 2: Empirical distribution functions of durations of rule of Roman Emperors (steps) with corresponding exponential approximations (dashed lines) and compensators $K(t, F_n)$ (lines with asterisks): (a) chronology "K", (b) chronology "P", (c) chronology "G".

have been recognised as Emperor and from what date. We offer no opinion on such matters, but as we noted above, our conclusions are not affected by the choice of chronology.

In Fig. 2 (a) we show the empirical distribution function of the rule lengths calculated from "K", along with the approximating exponential distribution function and the graph of the compensator (2). The value of the Kolmogorov-Smirnov statistic is $D_n = 0.208\sqrt{53} = 1.52$, which corresponds to p-value of 0.26. Thus there is no evidence against the hypothesis of exponentiality. The situation with data from "P" is very similar, as shown in Fig. 2 (b). The value of the Kolmogorov-Smirnov statistic is now $D_n = 0.225\sqrt{59} = 1.73$ with p-value 0.17. Finally, agreement between data from "G" and the exponential distribution is also good, as can be seen from Fig 2 (c). The value of the Kolmogorov-Smirnov statistic is $D_n = 0.229\sqrt{52} = 1.65$, with a corresponding p-value of 0.20.

For our investigation of Chinese Emperors, we used the chronology given in Encyclopaedia Britannica (2002, "China", vol. 3, p. 222). We started with the Tung (Eastern) Chou dynasty, with the first Emperor being Chi I-chia (P'ing-wang) (770-719 BC) and ended with Chu Yu-chien (1627-1644 AD), the last Emperor of the Ming dynasty. The total number of names that we analysed is $n = 367$. Agreement between the data and a fitted exponential distribution function looks very good (Fig. 3). However, the Kolmogorov-Smirnov test rejects exponentiality, with $D_n = 0.128\sqrt{367} = 2.46$ and a p-value of 0.028. From other considerations we know that exponentiality here should be rejected, but we will discuss this in more detail elsewhere. Next we consider competing explanations for the observed exponential patterns we have illustrated so far.

4 Possible explanations for the exponential pattern

In the case of rulers from long ago, like the Roman or Chinese Emperors above, one could argue, perhaps, that their lives were constantly exposed to turmoil and life threatening challenges, no matter whether in peace or at war. Therefore we might expect phenomena similar to deaths from accidents occurring 'routinely' and hence, see exponential distributions of the durations of reign. This sort of reasoning does not seem to us sufficient. Why should these life threatening challenges, plots and uprisings occur completely spontaneously and not as the result of the accumulation of controversies and tensions? Or, to put it differently, if there are many challenges and threats to a reign, usually there are also strong means

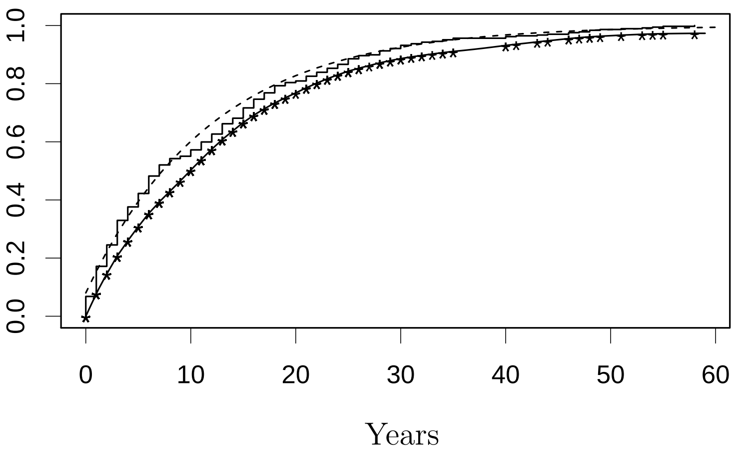

Figure 3: Empirical distribution function (steps) of durations of rule of Chinese Emperors and exponential approximation (dashed line). The line with asterisks gives the graph of the compensator $K(t, F_n)$.

to protect the reign and the ruler. Therefore only exceptionally strong challenges could stop the reign and overthrow the ruler. To justify the latter interpretation we note that in probability theory, typically the occurrences of unusually high levels of a stationary random process form a Poisson process (e.g., see Cramér and Leadbetter, 1967 or Resnik, 1987). Therefore, inter-occurrence times between exceptionally strong challenges would have an exponential distribution.

There is another, much simpler and trivial possible explanation for the exponential pattern we observe, which is the following. Before acquiring the title and position of Emperor, a person may typically have had a long political or military career and therefore would be of mature age. The distribution function of the remaining lifetime of a person of age X is the rescaled tail distribution

$$F(t|X) = \frac{F(X+t) - F(X)}{1 - F(X)}$$

of the distribution function F of lifetimes in the general population. This tail distribution (e.g., see Fig. 4) is indeed a concave function of t for reasonably large X, thus resembling an exponential distribution. The mixture of distribution functions $F(\cdot|X)$ over different ages of ascent may also be a distribution function similar in shape to an exponential distribution, and it may only be this similarity that we are observing.

According to the first explanation, the nature and statistical properties of the remaining life of an Emperor was changing at ascent in a very clear way – the remaining life became more haphazard and vulnerable. According to the second explanation, ascent to Emperor did not affect the remaining life distribution of the person, or affected it only a little.

To obtain a first numerical impression of the problem, we considered as an example the empirical distribution function of the life-times of New Zealand males, which show the median age at death equal to 80 years. We scaled the distribution down to make the median age at death smaller – firstly equal to 60 years, then 50 years then 40 years (multiplicative factors of $c = 0.75$, $c = 0.625$ and $c = 0.5$). Then, using the ages at ascent given in "P",

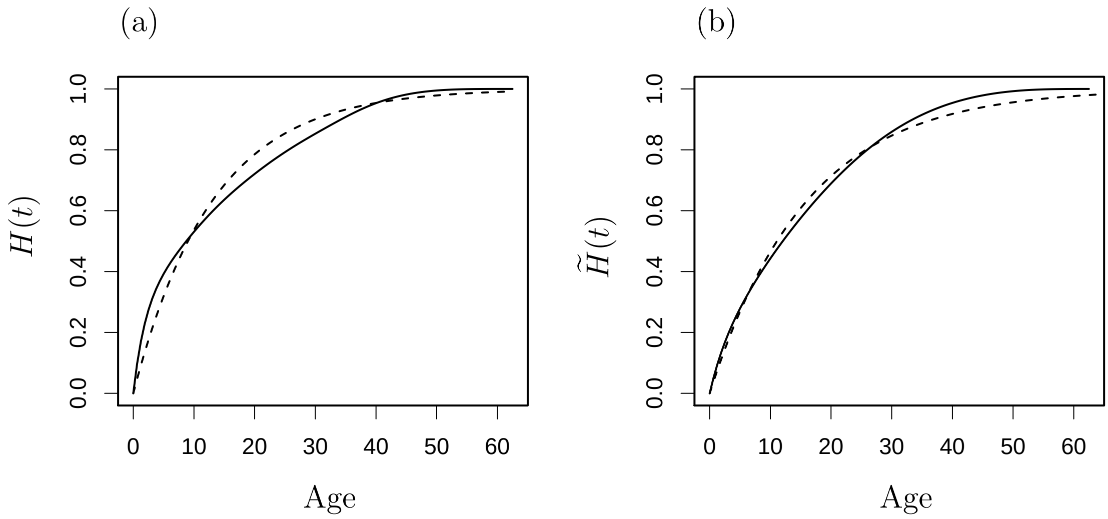

Figure 4: Mixtures of the remaining life distribution of NZ males (solid lines): (a) using ages at ascent from "P", (b) using a normal mixing distribution. In both cases the approximating exponential distribution is shown (dashed lines).

$X_i, i = 1, \ldots, n$, we calculated the mixed remaining life distribution

$$H(t) = \frac{1}{n} \sum_{i=1}^{n} F(t|X_i)$$

of the n remaining life distributions. In all cases it was indeed a concave function of t, but still different from an exponential distribution. The graph of $H(\cdot)$ with median age at death of 50 years, which was the closest fit to an exponential distribution of those considered, is given in Fig. 4 (a).

Instead of using empirical values for mixing ages one could use, for example, a fitted normal distribution truncated at ages 0 and 80, and consider

$$\widetilde{H}(t) = \int_0^{80} \frac{F(t|x)\Phi_{(\mu,\sigma)}(dx)}{[\Phi_{(\mu,\sigma)}(80) - \Phi_{(\mu,\sigma)}(0)]},$$

where $\Phi_{(\mu,\sigma)}(x)$ denotes the normal distribution function. The parameters of the fitted normal distribution were $\mu = 40$ years and $\sigma = 18$ years. Fig. 4 (b) shows that like H, $\widetilde{H}$ also resembles an exponential distribution function. While there are differences in both cases, these differences are relatively small. Therefore, it is a difficult statistical problem to distinguish between an exponential distribution and a mixed remaining life distribution, especially when the number of observations is not large.

5 Further analysis and some European monarchies

A sensible next step is to consider the rate of 'mortality' of the reign times (i.e., the failure rate). In Fig. 5, for the reign times that we consider from "P", we show the empirical failure rate and the corresponding fitted $\hat{\alpha}$, assuming an exponential distribution. The empirical failure rate indeed appears as a random function that fluctuates about a constant level – just as expected in the case of an exponential distribution. However, conclusive analysis based on the failure rate requires somewhat larger sample sizes (e.g., see Andersen *et al.*, 1993).

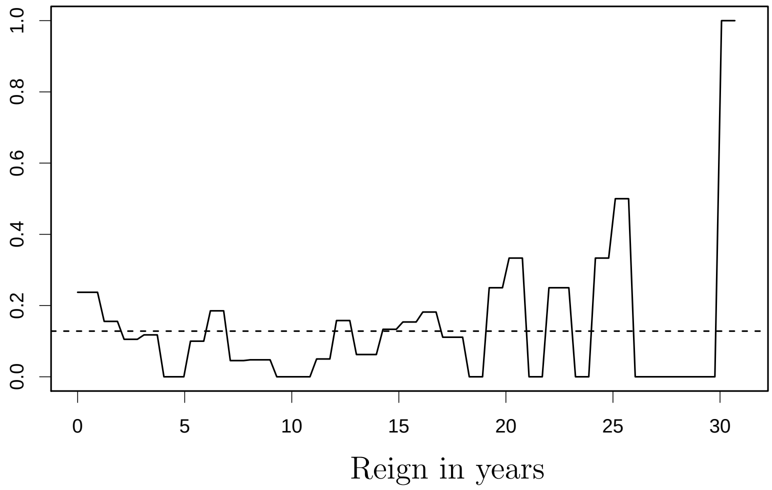

Figure 5: Empirical failure rate of reign times for Roman Emperors from “P” (solid line) and corresponding fit from an exponential distribution (dashed line).

Data on ages of ascent (from “P”) had a very wide range, in fact more or less agreeing with a uniform distribution on, say, 0 to 80 years. Further, as noted above, the data agrees well with a truncated normal distribution with a mean of 40 years and a (large) standard deviation of 18 years; see Fig. 6. In other words it is incorrect to think that emperors were typically persons of mature age; many of them were middle aged or young and some of them were very young. Note that here, and below, we excluded eight Emperors (Aemilianus, Valerian, Quintillus, Florianus, Diocletian, Severus II, Maxentius and Maximinus Daia) because of a large uncertainty in their possible ages at ascent.

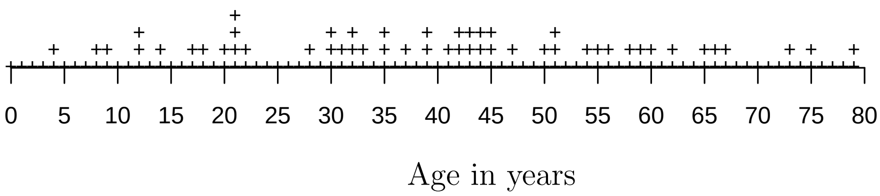

Figure 6: Ages at ascent of Roman Emperors from “P”.

So it is natural to ask whether Emperors who started to reign at a young age lived as long as could be expected for a ‘common’ member of the population with the same age. Fig. 7 (a) shows age of ascent and the duration of reign for each Emperor. Along with the data we show the expected remaining life at each age, as taken from Ulpian’s Tables (see, e.g., Haberman and Sibbett (1995) or Frier (1982), who argues that the Tables may give good estimates of the median remaining life, rather than the average). Visually it is quite clear that younger Emperors lived much shorter lives than the Tables suggest – not a single data point exceeds the expected level for ages at ascent below 33 years.

There are differing views on the accuracy of Ulpian’s Tables for expected remaining life (e.g., see Kopf, 1927). One can proceed without any reference to these tables though. We have seen that based solely on the durations of rule it is difficult to distinguish between our two explanations for observed exponential patterns, because they lead to distributions

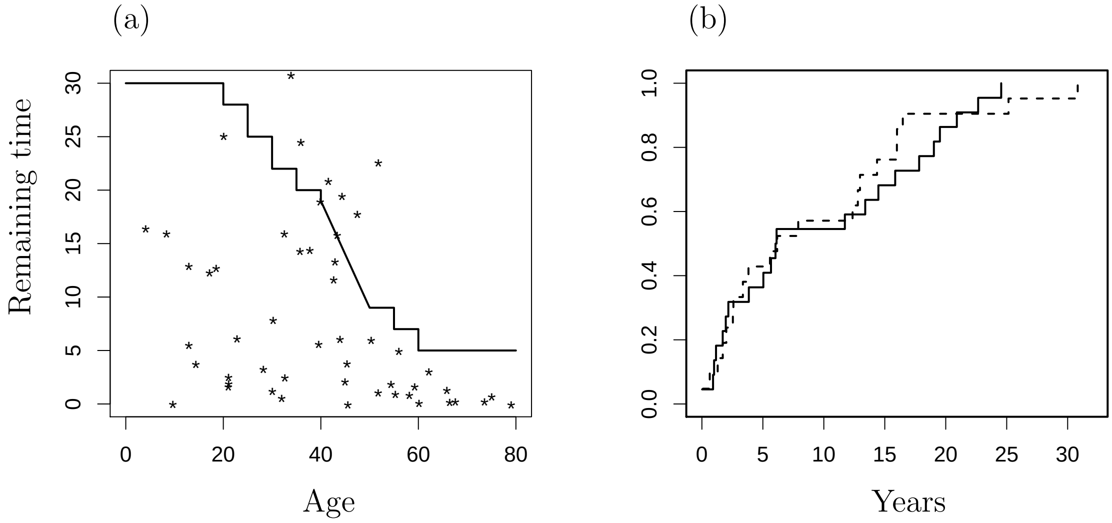

Figure 7: (a) Rule length of Roman Emperors by age at ascent (points) and expected remaining life from Ulpian's Tables by age (line), (b) empirical distribution functions of rule lengths for 'old' (solid line) and 'young' (dashed line) Roman Emperors.

that are close to each other. However we can now make further use of the age at ascent as an additional variable. If rule lengths are closely related to remaining lifetimes, then the younger the age of ascent the longer a person should rule (and live). Alternatively, with genuine exponentiality of rule lengths, the age of ascent should be independent of rule length.

To use relevant statistical tests we first excluded the group of eight Emperors, visible in Fig. 7 (a), whose ages at ascent were 60 or more. For this group of people the hardship of being Emperor was combined with relatively old age, and it is natural that they did not live (or rule) long. We divided the remaining $n = 43$ observations into two groups at the median: those who ascended at age ≤ 35 ('young') and at age > 35 ('old'); Fig. 7 (b) shows their empirical distribution functions. They look very convincingly as coming from the same underlying distribution, and there is no real need for a formal test. Nevertheless, the value of the two sample Kolmogorov-Smirnov statistic is 0.582, with a p-value of 0.79. Further, Spearman's rank correlation between age at ascent and rule length is estimated as $r_s = -0.15$; a lower tailed test that the rank correlation is zero has a p-value of 0.17. Hence there is no evidence of a lack of independence between age at ascent and rule length, and the hypothesis of exponential lengths of rule seems well supported.

It would be nice to have an example of rule lengths which do not display the pattern of exponentiality. To find such an example we considered data from some European monarchies of later times (from the 15th century to the early 20 century), for which both rule lengths and ages at ascent are available. One might expect greater stability in more recent times, giving more stable reigns and hence a lack of exponentiality for lengths of rule.

As a first example, consider the durations of rule of $n = 23$ Spanish monarchs, starting with Isabella I in 1474. Fig. 8 (a) shows the empirical distribution function and the fitted exponential distribution. A goodness of fit test does not reject the hypothesis of exponentiality, with $D_n = 0.395\sqrt{23} = 1.89$ and a p-value of 0.12. However, independence between age at ascent and rule length is certainly rejected. As before, a median-split of ages at ascent was made into 'young' and 'old' groups, as shown in Fig. 8 (b); the corresponding two sample Kolmogorov-Smirnov statistic was 1.56, with a p-value of 0.015. Spearman's rank correlation between age at ascent and rule length is estimated as $r_s = -0.46$, with a lower-tailed p-value of 0.013; see Fig. 8 (c). So it appears that younger Spanish monarchs did indeed rule for

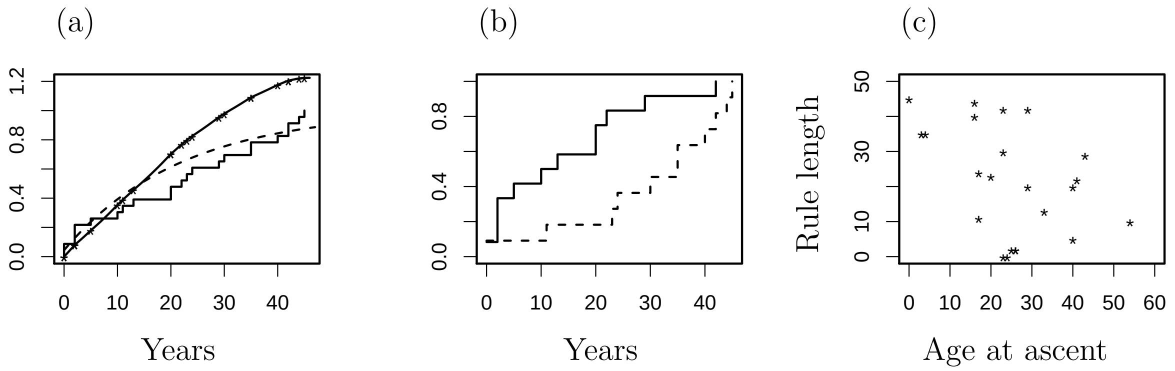

Figure 8: Spanish monarchs: (a) Empirical distribution function of durations of rule (steps) with corresponding exponential approximation (dashed line) and compensator $K(t, F_n)$ (line with asterisks), (b) comparison of empirical distribution functions of rule lengths for 'old' (solid line) and 'young' (dashed line), (c) rule length by age at ascent.

longer, and hence rule lengths as a whole are not exponential. Note this happens despite the fact that in a number of cases the end of reign was an abdication rather than the end of life, and also in two cases the same person ruled twice, but we counted these periods separately.

For our second example we look at data from the Romanov dynasty of Russia, starting with Ivan IV in 1547 ($n = 24$). Fig. 9 (a) shows the empirical distribution function and the fitted exponential distribution. This data agrees with exponentiality much more closely than the Spanish monarchs, with $D_n = 0.244\sqrt{24} = 1.20$ and a p-value of 0.46. Further, independence between age at ascent and rule length is not rejected. A median-split into 'young' and 'old' ages at ascent, as shown in Fig. 9 (b), gives a two sample Kolmogorov-Smirnov statistic of 0.61, with a p-value of 0.87. Spearman's rank correlation between age at ascent and rule length is estimated as $r_s = -0.12$, with a lower-tailed p-value of 0.29; see Fig. 9 (c). Note that there was some turmoil on the throne and forced abdications in the history of this dynasty. The agreement with exponentiality would be even closer if we included data starting from the time of Ivan the Terrible.

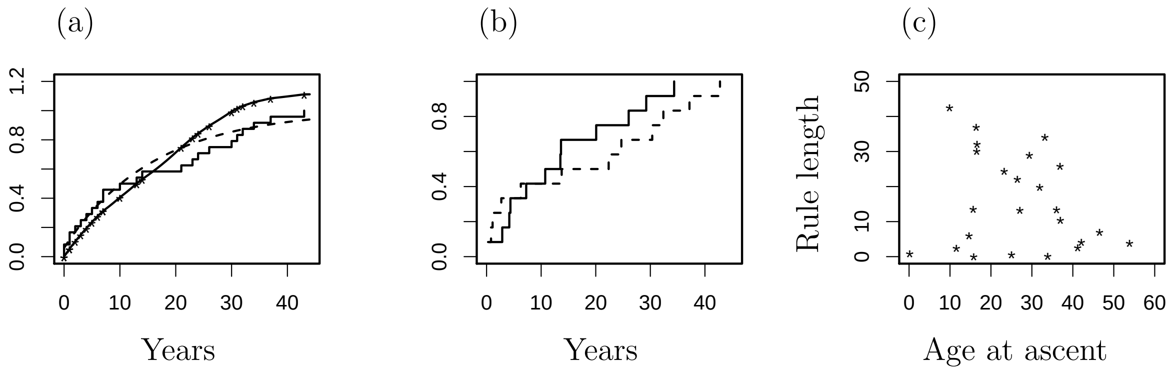

Figure 9: Russian rulers: (a) Empirical distribution function of durations of rule (steps) with corresponding exponential approximation (dashed line) and compensator $K(t, F_n)$ (line with asterisks), (b) comparison of empirical distribution functions of rule lengths for 'old' (solid line) and 'young' (dashed line), (c) rule length by age at ascent.

A final example also leads to rejection of exponential rule lengths, but less conclusively than for Spain. We consider $n = 22$ British monarchs, from Henry VII (reign: 1483-1509) to George VI (1936-1952). Fig. 10 (a) shows the empirical distribution function and the fitted exponential distribution. The complete data shows good agreement with an exponential pattern, since $D_n = 0.219\sqrt{22} = 1.03$ with a p-value of 0.60. A median-split into 'young' and 'old' ages at ascent gives a two sample Kolmogorov-Smirnov statistic of 1.09 with p-value 0.19; see Fig. 10 (b). While those two tests are consistent, Figs. 10 (b) and (c) suggest that younger monarchs ruled longer. That is confirmed by Spearman's rank correlation between age at ascent and rule length, estimated as $r_s = -0.46$ with a lower-tailed p-value of 0.015.

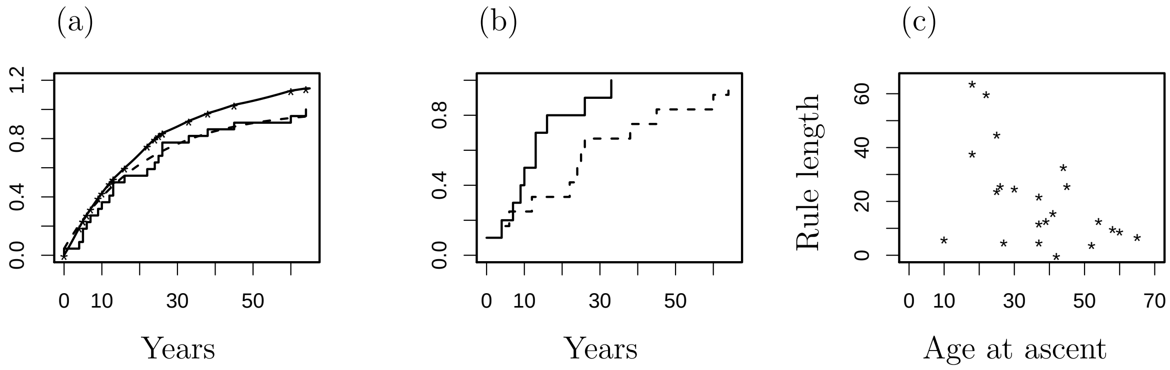

Figure 10: British monarchs: (a) Empirical distribution function of durations of rule (steps) with corresponding exponential approximation (dashed line) and compensator $K(t, F_n)$ (line with asterisks), (b) comparison of empirical distribution functions of rule lengths for 'old' (solid line) and 'young' (dashed line), (c) rule length by age at ascent.

6 Remarks in conclusion

As soon as the process of aging is involved in the 'life' of a mechanism or a biological organism, its survival time can not be exponential. We believed, a priori, that the same must be true for the duration of rule – the rule stops as a result of the accumulation of political, economic or personal tensions.

Contrary to this, the durations of rule of Roman Emperors for the period of 'decline and fall' exhibit very good agreement with an exponential distribution. Hence we must infer that reigns stopped in unexpected and unpredictable ways. This remarkably brittle nature of rule could be thought of as a statistical manifestation of the term 'decline and fall'. However, we show that an exponential pattern is to be found in a much wider range of situations – not only for a wider period of Roman history, e.g., starting from Augustus Octavian, but also in a very long span of Chinese history. For European monarchies of more recent times though, say from the 15th century, this is typically not the case and the reign is more stable.

To test exponentiality we used asymptotically distribution-free statistics, based on the transformed empirical process studied recently in Haywood and Khmaladze (2005). Our hypothesis was that the act of election, or pronouncement, of a person to Emperor influenced the duration of their remaining life: this remaining life became independent of the age of ascent and the duration of rule had an exponential distribution. All these durations were from the same distribution with a scale parameter which characterized the epoch. This hypothesis goes somewhat against the approach of some published research in Roman demography (e.g., Scheidel, 1999).

Statistically, the problem was one of testing against local alternatives – the natural alternative we had to consider was of a clearly ‘local’ character, but with no simple analytic representation. The goodness of fit test for exponentiality was complemented by some other well known techniques, which used the additional information of age at ascent to ruler.

Appendix

A Gibbon Chronology

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B.C. A.D.
 27-14     Augustus
    14-37  Tiberius
    37-41  Gaius (Caligula)
    41-54  Claudius
    54-68  Nero
    68-69  Galba
       69  Otho
       69  Vitellius
    69-79  Vespasian
    79-81  Titus
    81-96  Domitian
    96-98  Nerva
   98-117  Trajan
  117-138  Hadrian
  138-161  Antoninus Pius
  161-180  Marcus Aurelius with Lucius Verus 161-169 and Commodus from 177
  180-192  Commodus
      193  Pertinax followed by Didius Julianus
  193-211  Septimius Severus with Caracalla from 198 and Geta from 209
  211-217  Antonius (Caracella) with Geta (211-212)
  217-218  Macrinus with Diadumenianus in 218
  218-222  Antonius (Elagabalus)
  222-235  Severus Alexander
  235-238  Maximinus Thrax
      238  Gordian I, Gordian II, Pupienus (Maximus) and Balbinus
  238-244  Gordian III
  244-249  Philip the Arab with his son Philip 247-249
  249-251  Decius
  251-253  Trebonianus Gallus and Volusianus
  253-260  Valerian with Gallienus
  260-268  Gallienus
  268-270  Claudius II Gothicus
  270-275  Aurelian
  275-276  Tacitus (and Florianus 276)
  276-282  Probus
  282-283  Carus
  283-284  Carinus and Numerian
  284-305  Diocletian }
  286-305  Maximian   } Both abdicated in 305
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305-311	Galerius, associated with him over various periods Constantius I Chlorus, Severus II, Licinius, Constantine I, and Maximinus Daza. In 309 there were six Augusti
311-324	Constantine I and Licinius
324-337	Constantine I
337-340	Constantine II, Constantius II and Constans
340-350	Constantius II and Constans
350-361	Constantius II
361-363	Julian
363-364	Jovian
364-375	Valentinian I and Valens with Gratian from 367
375-378	Valens, Gratian and Valentinian II
378-395	Theodosius I (the Great) reigned with Gratian and Valentinian II from 378-383, with Valentinian II and Arcadius 383 to 392, and with Arcadius and Honorius 392 until his death in 395

B Kienast Chronology

Augustus (16. Jan. 27 v. Chr.-19. Aug. 14 n. Chr.)
Tiberius (19. Aug. 14-16. Marz 37)
Caligula (18. Marz 37-24. Jan. 41)
Claudius (24. Jan. 41-13. Okt. 54)
Nero (13. Okt. 54-9. Juni 68)
Galba (8. Juni 68-15. Jan. 69)
Otho (15. Jan.-16. April 69)
Vitellius (2. Jan.-20 Dez. 69)
Vespasian (1. Juli 69-23 Juni 79)
Titus (24. Juni 79-13. Sept. 81)
Domitian (14. Sept. 81-18. Sept. 96)
Nerva (18. Sept. 96-27.[?] Jan. 98)
Trajan (28. Jan. 98-7. Aug. 117)
Hadrian (11. Aug. 117-10. Juli 138)
Antoninus Pius (10. Juli 138-7. Marz 161)
Mark Aurel (7. Marz 161-17. Marz 180)
Commodus (17. Marz 180-31. Dez. 192)
Pertinax (31. Dez. 192-28. Marz 193)
Didius Iulianus (28. Marz-1. Juni 193)
Septimius Severus (9. April 193-4. Febr. 211)
Caracalla (4. Febr. 211-8. April 217)
Macrinus (11. April 217-8. Juni 218)
Elagabal (16. Mai 218-11. Marz 222)
Severus Alexander (13. Marz 222-Febr./Marz 235)
Maximinus Thrax (Febr./Marz 235-Mitte April [?] 238)
Gordian I. (Jan [?] 238)
Gordian II. (Jan [?] 238)
Pupienius (Ende Jan./Anf. Febr. [?]-Anf. Mai [?] 238)
Balbinus (Jan./Febr. [?]-Mai [?] 238)
Gordian III. (Jan./Febr. [?] 238-Anf. 244)
Philippus Arabs (Anf. 244-Sept./Okt. 249)
Decius (Sept./Okt. 249-Juni 251)
Trebonianus Gallus (Juni [?] 251-Aug. [?] 253)

Aemilius Aemilianus (Juli/Aug.-Sept./Okt. 253)
Valerian (Juni/Aug. 253-Juni [?] 260)
Gallienus (Sept./Okt. 253-ca. Sept. 268)
Claudius II. Gothicus (Sept./Okt. 268-Sept. 270)
Quintillus (September 270)
Aurelian (Sept. 270-Sept./Okt. 275)
Tacitus (Ende 275-Mitte 276)
Florianus (Mitte-Herbst 276)
Probus (Sommer 276-Herbst 282)
Carus (Aug./Sept. 282-Juli/Aug. 283)
Numerianus (Juli/Aug. [?] 283-Nov. 284)
Carinus (Fruhjahr 283-Aug./Sept. 285)
Diocletian (20. Nov. 284-1. Mai 305)
Maximian (Okt./Dez. 285-ca. Juli 310)
Constantius I. (1. Marz 293-15. Juli 306)
Galerius (21 Mai [?] 293-Anf. MAi 311)
Maximinus Daia (1. Mai 305-Spatsommer 313)
Severus II. (1. Mai 305-Marz/April 307)
Maxentius (28. Okt. 306-28. Okt. 312)
Licinius (11. Nov 307-19 Sept. 324)
Constantin I. (25. Juli 306-22. Mai 337)
Constantin II. (9. Sept. 337-Anf. April 340)
Constans (9. Sept. 337-18. Jan. 350)
Constantius II. (9. Sept. 337-3. Nov. 361)
Julian (ca. Febr. 360-26./27. Juni 363)
Jovian (27. Juni 363-17. Febr. 364)
Valentinian I. (25. Febr. 364-17. Nov. 375)
Valens (28. Marz 364-9. Aug. 378)
Gratian (24. Aug. 367-25. Aug. 383)
Valentinian II. (22. Nov. 375-15. Mai 392)
Theodosius I. (19. Jan. 379-17. Jan. 395)

C Parkin Chronology

Augustus
23 Sept. 63 BC
16 Jan. 27 BC, aged 35
19 Aug. AD 14, aged 75 yrs

Tiberius
16 Nov. 42 BC
19 Aug. AD 14, aged 54
16 (26 Dio?) March 37, aged 77

Caius
31 August AD 12
18 (26 Dio?) March 37, aged 24
24 (22?) Jan. 41, aged 28

Claudius
1 August 10 BC
24 Jan. AD 41, aged 49
13 Oct. 54, aged 63

Nero
15 Dec. 37
13 Oct. 54, aged 16
9 June 68, aged 30

Galba
24 Dec 3 BC (probably, rather than 5 BC)
8 June 68, age 69
15 Jan 69, age 70

Otho
28 April 32
15 Jan 69, aged 36
16 April 69, aged 36 (few weeks short of 37)

Vitellius
7th or 24th of Sept. AD 12 or 15 (7/9/12 better)
2 Jan 69, aged 53 or 56
20/21 Dec 69, aged 54 or 57

Vespasian
9th or 14th or 17th Nov. AD 9
1 July 69, aged 59
23 June 79, aged 69 [some would say 24th June]

Titus
30 Dec 39
24 June 79, aged 39
13 Sept. 81, aged 41

Domitian
24 Oct 51
14 Sept 81, aged 29
18 Sept 96, aged 44

Nerva
8 Nov. 30 (or 35 ?)
18 Sept 96, aged 60 or 65
27/28 Jan. 98, aged 62 or 67

Trajan
18 Sept. 53 (56?)
28 Jan 98, aged 44 (41?)
7/8 (11?) Aug. 117, aged 63 (60?)

Hadrian
24 Jan 76
11 Aug 117, aged 41
10 July 138, aged 62

Antoninus Pius
19 Sept 86
10 July 138, aged 51
7 March 161, aged 74 years 5 months 18 days

Marcus Aurelius
26 April 121
7 March 161, aged 39
17 March 180, aged 58 years, 11 months

Lucius Verus
15 Dec 130
7 March 161, aged 30
Jan or Feb 169, aged 38

Commodus
31 Aug 161
late 176 / early 177, aged 15. OR 17 March 180 (death of Aurelius), aged 18
31 Dec. 192, aged 31

Pertinax
1 August 126
31 Dec 192, aged 66
28 March 193, aged 66

Didius Iulianus
30 Jan 133
28 March 193, aged 60
1 June 193, aged 60

Septimius Severus
11 April 145 or 146
9 April 193, aged 46 or 47 (2 days short of birthday)
4 Feb 211 aged 64 or 65

Clodius Albinus
25 November 147
April 193, aged 45?
19 Feb 197, aged 49?

C. Pescennius Niger (Iustus)
AD 135-140 ?
April 193, aged 53-58?
April 194, aged 54-59?

Caracalla
4 April 188 (186 ?)
28 Jan or 8/9 April 198, aged 9/10 (11/12 ?) OR 4 Feb 211, aged 22 (24 ?)
8 April 217, aged 29 (31 ?)

Geta
7 March 189
Sept or Oct 209 or 210, aged 20/21
Dec 211, aged 22

Macrinus
164 or 166
11 April 217, aged 52/53 or 50/51
8 June 218, aged 53/54 or 51/52

Diadumenianus
14 Sept 208
May 218, aged 9
June 218, aged 9

Elagabalus (Heliogabalus)
ca. 203/4
16 May 218, aged 13-15 (?)
11 March 222, aged 17-19 (?)

Severus Alexander
1 Oct 208 or 209
13 March 222, aged 12/13
Feb or March 235, aged 25/26

Maximinus Thrax
172/3
Feb or (mid-?)March 235, aged 61-63
April (early June?) 238, aged 64-66

Gordian I
158/9 (a little later) ? (ca. 178 ?)
Jan (mid/late March?) 238 ?, aged 78-80?
20 Jan (late April?) 238?, aged 78-80?

Gordian II
ca. 192
Jan (mid/late March?) 238, aged ca. 45
20 Jan (late April?) 238, aged ca. 45/46 ?

Balbinus
? (Kienast p.193: 'unbekannt und nicht zu ermitteln')
Jan or Feb (late April?) 238
May (early August?) 238

Pupienus
ca, 164 ?
Jan or Feb (late April?) 238, aged 74 (?)
May (early August?) 238, aged 74 (?)

Gordian III
20 Jan 225 or 226
238, aged 11-13 years
early 244, aged 18/19

Philippus
204 ??
early 244, aged 39/40 ??
Sept or Oct 249, aged 44/45 ??

Decius
190 or 200 ?
? Sept or Oct 249, aged 48/49 or 58/59 ?
June 251, aged 50/51 or 60/61 ?

Trebonianus Gallus
206?
June 251 ?, aged 44/45?
Aug 253 ?, aged 46/47?

Volusianus (son of Trebonianus Gallus)
ca. 230
August ? 251, aged 20/21
August ? 253, aged 22/23

Aemilianus
ca. 207 or 214 ?
July or Aug 253, aged 39/39 or 45/46 ?
Sept or Oct 253, aged 38/39 or 45/46 ?

Valerian
ca. 193 or 200 ?
ca, July 253, aged 53/60 ?
taken prisoner ca. June 260, aged 60/67 ?
after 262, aged 62+/69+ ?

Gallienus
ca. 213 or 218
Sept or Oct 253, aged 39/40 or 34/35
22 March, or in Sept., 268, aged 54/55 or 49/50

Claudius II Gothicus
10 May 214 ?
Sept or Oct 268, aged 54?
Sept 270, aged 56?

Quintillus (brother of Claudius II)
?
Sept 270
Sept 270

Aurelian
9 Sept 214 ?
Sept 270, aged 56?
Sept or Oct 275, aged 61?

Tacitus
ca. Sept 200 ?
late Sept 275, aged 74/75?
mid 276, aged 75/76

Florianus (brother of Tacitus)
?
mid 276
autumn 276

Probus
19 Sept 232
summer/autumn 276, aged 43/44
autumn 282, aged 49/50

Carus
ca. 224 ?
Aug or Sept 282, aged 58?
July or Aug 283. aged 59?

Carinus (older [?] son of Carus)
ca. 250
early 283, aged 32/33?
Aug or Sept 285, aged 34/35?

Numerianus (younger [?] son of Carus)
ca. 253
July or Aug 283, aged 29/30?
Nov 284, aged 30/31?

Diocletian
22 Dec., 236/7 or 244 or 245 or 247/8 ?? (or even as early as ca. 225 ?!)
17 or 20 Nov 284, aged 35-47 (if 245, then aged 38)
abdicated 1 May 305, aged 56-68 (if 245, then aged 59)
3 Dec 311 or 312 or 313 or 315 or 316, aged 63-79 (if 245, then aged 67 in 313)

Maximian
21 July (nicht 22 Dec, says Kleinast) 249/250
Oct or Dec 285, or 1 April 286, aged 35/36
abdicated 1 May 305, aged 54/555
back again late 306, aged 56/57
ca. July 310, aged 60/61

Constantius I
31 March ca. 250
1 March 293, or 1 May 305, aged ca. 43/55
25 July 306, aged ca. 56

Galerius
250s
21 May 293 or 1 May 305, aged 33-43 or 45-55 ?
May 311, aged 51-61 ?

Severus II
?
1 May 305 or 306
March or April 307

Maxentius
ca. 275-287
28 Oct 306, aged 18-31
28 Oct 312, aged 24-37

Constantine
27 Feb 272 or 273 (ca. 280, Nixon)
25 July 306, aged 33/34
22 May 337, aged 64/65

Licinius
ca. 265
11 Nov 308, aged ca. 43
resigned 19 Sept 324, aged ca. 59
spring 325, aged ca. 60

Maximinus Daia
20 Nov ca. 270 or 285
1 May 305 or 308 or 309 or 310, aged 19/34 or 22/37 or 23/38 or 24/39
mid 313, aged 27/42

Constantine II
7 August 316 OR Feb 217 ?
9 Sept 337, aged 20/21
early April 340, aged 23

Constans I
320 OR 323 ?
9 Sept 337, aged 14 or 17
ca. 18 Jan 350, aged 27 or 30

Constantius II
7 Aug 317
9 Sept 337, aged 20
3 Nov 362, aged 44

Julian
May or June, ca. 331 or 332
ca. Feb 360, aged 29?
26 or 27 June 363, aged 32?

Jovian
331
27 June 363, aged 32
17 Feb 364, aged 33

Valentinian I
321
25 Feb 364, aged 42/43
17 Nov 375, aged 54

Valens
ca. 328
28 March 364, aged 35?
9 Aug 378, aged 49?

Gratian
18 April 359
24 Aug 367, aged 8
25 Aug 383, aged 24

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Valentinian II
autumn (not 2 July) 371
22 Nov 375, aged 4
15 May 392, aged 20

Theodosius I
11 Jan 346 or 347
19 Jan 379, aged 32/33
17 Jan 395, aged 48/49
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