Pólya - Aeppli of order k Risk Model

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Abstract

In this study we define the Pólya - Aeppli process of order k as a compound Poisson process with truncated geometric compounding distribution with success probability $1-\rho > 0$ and investigate some of its basic properties. Using simulation we provide a comparison between the sample paths of the Pólya - Aeppli process and the Poisson process. Also, we consider a risk model in which the claim counting process $\{N(t)\}$ is Pólya - Aeppli process of order k, and call it a Pólya - Aeppli of order k risk model. For the Pólya - Aeppli of order k risk model we derive the joint distribution of the time to ruin and the deficit at ruin as well as the ruin probability. We discuss in detail the particular case of exponentially distributed claims and provide simulation results for more general cases.

Key words: Distributions of order k, Pólya - Aeppli process, ruin probability, simulation, sensitivity analysis

AMS subject classifications: 60K10; 62P05.

1 Introduction

Assume that the standard model of an insurance company, called risk process $\{X(t), t \geqslant 0\}$ is given by

$$X(t) = ct - \sum_{i=1}^{N(t)} Z_i \quad (\sum_1^0 = 0). \tag{1}$$

Here c is a premium income per unit time, $N(t)$ is the counting process, $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent identically distributed, positive random variables, independent of $N(t)$ with Z_i representing the ith claim amount. We assume that the individual claim amounts have a continuous distribution with distribution function F, $F(0) = 0$, and mean value $\mu = EZ_1 < \infty$. In the classical risk model the process $N(t)$ is a homogeneous Poisson process, see for instance Grandell (1991), [6] and Rolski et all. (1999), [13]. The most popular generalization of homogeneity is a compound Poisson process.

In this paper we suppose that the counting process $N(t)$ is a compound Poisson process with discrete compounding distribution, i.e. $N(t) = \sum_{i=1}^{N_1(t)} Y_i$, where $Y_1, Y_2, \ldots$ are independent identically distributed random variables, independent of $N_1(t)$ and $N_1(t) \sim Po(\lambda t)$. Let Y denotes the compounding random variable with probability generating function (PGF) $P_Y(s) = Es^Y$. Then the PGF of the counting process is given by

$$P_{N(t)}(s) = e^{-\lambda t[1 - P_Y(s)]}. \tag{2}$$

In Section 2 we define the Pólya - Aeppli distribution of order k. The Pólya - Aeppli process of order k as a pure birth process is given in Section 3. In Section 4 we consider the Pólya - Aeppli of order k risk model and derive a differential equation for the joint distribution of the time to ruin and the deficit at the time of ruin and an expression for the ruin probability. The results are illustrated for the particular case of exponentially distributed claims. In Section 5, a simulation approach is reviewed and implemented for the risk model with exponential, gamma and Weibull distributed claims.

2 The Pólya - Aeppli distribution of order k

The discrete distributions of order k were introduced in early eighties in Philippou et all. (1983), [11], Philippou and Makri (1986), [12]. A good reference for the distributions of order k is the book of Balakrishnan and Koutras (2002), [3]. The main property of the distributions of order k is that they can be represented as Compound Generalized Power series distributions, where the compounding distribution is a discrete distribution over $k \geqslant 1$ points, see for example, Charalmbides (1986), [4] and Aki et all. (1984), [2].

All random variables considered in this study are assumed to be defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We consider a random variable N that is

$$N = Y_1 + Y_2 + \ldots + Y_{N_1}, \tag{3}$$

where $Y_1, Y_2, \ldots$ are mutually independent, non-negative, integer valued, identically distributed random variables, independent of N_1. The probability distribution of N is said to be a *compound distribution.* Let $P_Y(s)$, where Y is an arbitrary Y_i, be the common PGF of the sequence $Y_1, Y_2, \ldots$. We suppose that the random variable N_1 belongs to the family of Generalized Power series distributions (GPSD), see for example Patil (1962), [10] and Johnson et all. (2005), [7]. The PGF of the random variable N is given by

$$P_N(s) = \frac{g(\theta P_Y(s))}{g(\theta)}, \tag{4}$$

where $\theta > 0$ is a parameter and $g(\theta)$ is a series function. The random variable N is said to have a Compound GPSD. The distribution of Y is a *compounding distribution.*

Suppose that the random variable Y has a truncated geometric distribution with success probability $1-\rho$ and probability mass function (PMF) and PGF, given by

$$P(Y = m) = \frac{1-\rho}{1-\rho^k}\rho^{m-1}, \quad m = 1, 2, \ldots k \tag{5}$$

and

$$P_Y(s) = \frac{(1-\rho)s}{1-\rho^k}\frac{1-\rho^k s^k}{1-\rho s}, \tag{6}$$

where $k \geqslant 1$ is fixed integer number.

If $k \to \infty$, the truncated geometric distribution approaches the $Ge_1(1-\rho)$ distribution.

In this section we introduce the Pólya - Aeppli distribution of order k as a compound Poisson distribution with PGF given by

$$P_N(s) = e^{\lambda(P_Y(s)-1)}, \tag{7}$$

where $P_Y(s)$ is the PGF of the compounding distribution.

Definition 2.1 *The probability distribution defined by the PGF (7) and compounding distribution, given by (5) and (6) is called a Pólya - Aeppli distribution of order k with parameters $\lambda > 0$ and $\rho \in [0, 1)$, denoted by $PA_k(\lambda, \rho)$.*

3 The Pólya - Aeppli process of order k

The Pólya - Aeppli process of order k is introduced in Minkova (2010), [9]. It is a compound Poisson process with truncated geometric compounding distribution and PGF given by (2).

The second definition of the process is as pure birth process. Let $N(t)$ represents the state of the system at time $t \geqslant 0$. It is assumed that the process has state space $\mathcal{N}$, the non-negative integers. Let $\lambda > 0$ be any real number and $\rho \in (0, 1)$.

We assume that, for any small interval $h > 0$ the system state transition probabilities are as follows:

$$(8) \qquad P(N(t+h) = n \mid N(t) = m) = \begin{cases} 1 - \lambda h + o(h), & n = m, \\ \frac{1-\rho}{1-\rho^k} \rho^{i-1} \lambda h + o(h), & n = m + i, i = 1, 2, \ldots, k \end{cases}$$

for $m = 0, 1, \ldots$, where $o(h) \to 0$ as $h \to 0$. Note that this assumption implies that

$$P(N(t+h) = m + i \mid N(t) = m) = o(h), \quad \text{for} \quad i = k + 1, k + 2, \ldots,$$

Let $P_m(t) = P(N(t) = m),\ m = 0, 1, 2, \ldots$. Then Eq. (8) yields the following Kolmogorov forward equations:

$$(9) \qquad \begin{aligned} P_0'(t) &= -\lambda P_0(t), \\ P_m'(t) &= -\lambda P_m(t) + \tfrac{1-\rho}{1-\rho^k} \lambda \textstyle\sum_{j=1}^{m \wedge k} \rho^{j-1} P_{m-j}(t), \quad m = 1, 2, \ldots, \end{aligned}$$

with the the following initial conditions

$$(10) \qquad P_0(0) = 1 \quad \text{and} \quad P_m(0) = 0, \quad m = 1, 2, \ldots.$$

Let

$$h(u, t) = \sum_{m=0}^{\infty} u^m P_m(t)$$

be the PGF of the process $N(t)$. Multiplying the mth equation of (9) by u^m and summing over all $m = 0, 1, 2, \ldots$ we get the following differential equation

$$(11) \qquad \frac{\partial h(u, t)}{\partial t} = -\lambda[1 - P_Y(u)]h(u, t).$$

The solution of (11) with the initial condition $P_0(0) = 1$ is $h(u, t) = e^{-\lambda t[1-P_Y(u)]}$, which is the PGF of the $PA_k(\lambda t, \rho)$ distribution, given by (2) and (6).

Definition 3.1 *The counting process defined by (9) and (10) is called a Pólya - Aeppli process of order k.*

Remark 3.1 *In the case of $k \to \infty$, the Pólya - Aeppli process of order k, coincides with the Pólya - Aeppli process, defined in Minkova (2004), [8]. If $\rho = 0$, it is a homogeneous Poisson process.*

3.1 Poisson decomposition

Let us rewrite the PGF of $N(t) \sim PA_k(\lambda, \rho)$ in the following way:

$$\begin{aligned} P_{N(t)}(s) &= \exp\left(\lambda t \left[\tfrac{1-\rho}{1-\rho^k}(s + \rho s^2 + \ldots + \rho^{k-1}s^k) - 1\right]\right) \\ &= \prod_{i=1}^k e^{\lambda t \frac{1-\rho}{1-\rho^k}\rho^{i-1}(s^i - 1)}. \end{aligned}$$

The above means that $N(t)$ can be represented as a sum of k independent Poisson processes $M_1(t), \ldots, M_k(t)$ with means $EM_i(t) = \lambda \frac{1-\rho}{1-\rho^k}\rho^{i-1}t$ and PGFs $P_{M_i(t)}(s) = e^{\lambda t \frac{1-\rho}{1-\rho^k}\rho^{i-1}(s^i-1)}$, $i = 1, 2, \ldots, k$.

4 Application to Risk Theory

Consider the standard risk model $\{X(t),\ t \geqslant 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and given by (1). We consider the risk model (1), where $N(t)$ is Pólya - Aeppli process of order k and will call this process Pólya - Aeppli of order k risk model.

In this case the relative safety loading θ is defined by

$$\theta = \frac{EX(t)}{E\sum_{i=1}^{N(t)} Z_i} = \frac{c(1-\rho^k)}{\lambda\mu(1 + \rho + \rho^2 + \ldots + \rho^{k-1} - k\rho^k)} - 1.$$

In the case of positive safety loading $\theta > 0$, the premium income per unit time c should satisfy the following inequality

$$c > \frac{\lambda\mu(1 + \rho + \rho^2 + \ldots + \rho^{k-1} - k\rho^k)}{(1-\rho^k)}.$$

Let $\tau = \inf\{t : X(t) < -u\}$ with the convention of $\inf \emptyset = \infty$ be the time to ruin of an insurance company having initial capital $u \geqslant 0$. We denote by

$$\Psi(u) = P(\tau < \infty) \tag{12}$$

the ruin probability. Let $G(u, y)$ be the joint probability distribution of the time to ruin τ and the deficit in prior to ruin $D = |U(\tau)$, i.e.

$$G(u, y) = P(\tau < \infty, D \leqslant y). \tag{13}$$

and

$$\lim_{y \longrightarrow \infty} G(u, y) = \Psi(u). \tag{14}$$

Using the postulates (8), we get

$$G(u,y) = (1-\lambda h)\, G(u+ch,y) +$$

$$+\frac{1-\rho}{1-\rho^k}\lambda h \left[\int_0^{u+ch} G(u+ch-x,y)dF(x) + (F(u+ch+y) - F(u+ch))\right] +$$

$$+\frac{1-\rho}{1-\rho^k}\rho\lambda h \left[\int_0^{u+ch} G(u+ch-x,y)dF^{\star 2}(x) + \left(F^{\star 2}(u+ch+y) - F^{\star 2}(u+ch)\right)\right] +$$

$$\dots$$

$$+\frac{1-\rho}{1-\rho^k}\rho^{k-1}\lambda h \left[\int_0^{u+ch} G(u+ch-x,y)dF^{\star k}(x) + \left(F^{\star k}(u+ch+y) - F^{\star k}(u+ch)\right)\right] + o(h),$$

where $F^{\star m}(x), \quad m = 1, 2, \dots$ is the distribution function of $Z_1 + Z_2 + \dots + Z_m$. Rearranging the terms leads to

$$\frac{G(u+ch,y) - G(u,y)}{ch} = \frac{\lambda}{c} G(u+ch,y) -$$

$$-\frac{1-\rho}{1-\rho^k}\frac{\lambda}{c} \left[\int_0^{u+ch} G(u+ch-x,y)dF(x) + (F(u+ch+y) - F(u+ch))\right] -$$

$$-\frac{1-\rho}{1-\rho^k}\rho\frac{\lambda}{c} \left[\int_0^{u+ch} G(u+ch-x,y)dF^{*2}(x) + \left(F^{*2}(u+ch+y) - F^{*2}(u+ch)\right)\right] -$$

$$\dots$$

$$-\frac{1-\rho}{1-\rho^k}\rho^{k-1}\frac{\lambda}{c} \left[\int_0^{u+ch} G(u+ch-x,y)dF^{*k}(x) + \left(F^{*k}(u+ch+y) - F^{*k}(u+ch)\right)\right] + o(h)$$

Let

$$H(x) = \frac{1-\rho}{1-\rho^k}\left[F(x) + \rho F^{*2}(x) + \rho^2 F^{*3}(x) + \dots + \rho^{k-1} F^{*k}(x)\right]$$

be the non defective probability distribution function of the claims with

$$H(0) = 0, \qquad H(\infty) = 1.$$

By letting $h \to 0$ we obtain the following differential equation

$$\frac{\partial G(u,y)}{\partial u} = \frac{\lambda}{c}\left[G(u,y) - \int_0^u G(u-x,y)dH(x) - [H(u+y) - H(u)]\right] \tag{15}$$

and in terms of the safety loading

$$\frac{\partial G(u,y)}{\partial u} = \frac{1-\rho^k}{\mu(1+\rho+\dots+\rho^{k-1}-k\rho^k)}\frac{1}{1+\theta}\left[G(u,y) - \int_0^u G(u-x,y)dH(x) - [H(u+y) - H(u)]\right]. \tag{16}$$

4.1 Ruin probability

Theorem 4.1 *The probability of ruin $\Psi(u)$ satisfies the equation*

$$\frac{d\Psi(u)}{du} = \frac{\lambda}{c}\left[\Psi(u) - \int_0^u \Psi(u-x)dH(x) - [1-H(u)]\right], \quad u \geqslant 0. \tag{17}$$

Proof. The result follows from (15) and (14).

□

Theorem 4.2 *The function $G(0,y)$ is given by*

$$G(0,y) = \frac{\lambda}{c}\int_0^y [1-H(u)]du. \tag{18}$$

Proof. Integrating (15) from 0 to ∞ with $G(\infty, y) = 0$ leads to

$$-G(0,y) =$$

$$= \frac{\lambda}{c}\left[\int_0^\infty G(u,y)du - \int_0^\infty \int_0^u G(u-x,y)dH(x)du - \int_0^\infty (H(u+y) - H(u))du\right]$$

The change of variables in the double integral and simple calculations yield

$$G(0,y) = \frac{\lambda}{c}\int_0^\infty [H(u+y) - H(u)]du$$

and (18).

□

Theorem 4.3 *The ruin probability with no initial capital satisfies*

$$\Psi(0) = \frac{\lambda\mu}{(1-\rho)(1-\rho^k)c}[1-(k+1)\rho^k + k\rho^{k+1}]. \tag{19}$$

Proof. According to (18)

$$\Psi(0) = \lim_{y\to\infty} G(0,y) = \frac{\lambda}{c}\int_0^\infty [1-H(u)]du.$$

Let X be a random variable with distribution function $H(x)$. By the definition of $H(x)$ and $EZ = \mu$, we obtain

$$EX = \frac{\mu}{(1-\rho)(1-\rho^k)}[1-(k+1)\rho^k + k\rho^{k+1}].$$

Using the fact that $EX = \int_0^\infty [1-H(x)]dx$, we obtain (19).

□

Remark 4.1 *Based on (19), it is easy to see that the ruin probability with no initial capital dosn't depend on t.*

4.2 Exponentially distributed claims

Let us consider the case of exponentially distributed claim sizes, i.e., $F(x) = 1 - e^{-\frac{x}{\mu}}, \quad x \geqslant 0, \quad \mu > 0$. In this case, the function $h(x)$ is a mixture of Erlang density functions and is given by

$$h(x) = \sum_{i=1}^{k} q_i \frac{\left(\frac{x}{\mu}\right)^{i-1}}{\mu(i-1)!} e^{-\frac{x}{\mu}}, \ x > 0,$$

where $q_i = P(Y = i) = \frac{\rho^{i-1}(1-\rho)}{1-\rho^k}, \quad i = 1, 2, \ldots k$ is the mixing distribution, see Willmot and Lin (2001), [16]. For the survival function we obtain

$$\overline{H}(x) = \sum_{i=1}^{k} \overline{F}_Y(i-1) \frac{\left(\frac{x}{\mu}\right)^{i-1} e^{-\frac{x}{\mu}}}{(i-1)!}, \ x > 0, \tag{20}$$

where $\overline{F}_Y(0) = 1$ and

$$\overline{F}_Y(i-1) = P(Y > i-1) = \sum_{m=i}^{k} \frac{\rho^{m-1}(1-\rho)}{1-\rho^k} = \frac{\rho^{i-1}(1-\rho^{k-i+1})}{1-\rho^k}, \ i = 2, 3, \ldots, k.$$

Also, the survival function in (20) could be rewritten in the following form

$$\overline{H}(x) = \frac{e^{-\frac{x}{\mu}}}{1-\rho^k} \sum_{i=1}^{k} \frac{\left(\frac{\rho x}{\mu}\right)^{i-1}}{(i-1)!} [1 - \rho^{k-i+1}], \ x > 0.$$

In addition, the result of Theorem 4.2, in the case of exponentially distributed claims, modifies to

$$G(0, y) = \frac{\lambda \mu}{c} \sum_{i=0}^{k-1} \frac{\overline{F}_Y(i)}{i!} \gamma\left(i, \frac{y}{\mu}\right),$$

where $\gamma(i, x) = \int_0^x t^{i-1} e^{-t} dt$ is the incomplete gamma function.

5 Simulation

In what follows we briefly review a simulation approach for calculating the probability of ruin suggested in Dufresne and Gerber (1989), [5] and apply this approach for the case of exponentially distributed claims with initial capital $u = 0$. We confirm the validity of our simulation results by matching them with the analytical value of the ruin probability computed by using Eq. (19). In addition, we provide results for the case of non-zero initial capital not only for exponentially distributed claims but also for the case of gamma and Weibull claim distributions.

5.1 The background of the simulation approach

When it comes to computing the probability of ruin, the traditional approach of repeated simulations to observe the proportion of favourable outcomes is not applicable because of Eq. (12), i.e., it requires infinite simulation runs, which is impossible. An elegant and easy to implement approach for computing the ruin probability is proposed in Dufresne and Gerber (1989), [5], which is based on the following two facts:

- the probability of ruin is equivalent to the stationary distribution of a specific process, which is associated with the considered risk process;
- the stationary distribution of this associated process is computable by simulating its trajectories.

Next, we briefly summarise this approach. Denote by

(21) $$S(t) = \sum_{i=1}^{N(t)} Z_i \quad \text{and} \quad L(t) = S(t) - ct, \quad t \geqslant 0,$$

i.e., $L(t)$ is the accumulated loss at time t, and if

(22) $$M(t) = max_{0 \leqslant z \leqslant t} L(z),$$

then it is the maximum loss experienced over the interval $[0, t]$. The probability of survival to time t is equal to

(23) $$1 - \Psi(u, t) = P(M(t) \leqslant u),$$

i.e., it is given by the cdf of $M(t)$. Also, if $L = max_{t \geqslant 0} L(t)$, then the ruin probability is easily expressed in terms of L as $\Psi(u) = 1 - P(L \leqslant u)$. Now, let us consider the following associated process

(24) $$W(t) = L(t) - min_{0 \leqslant z \leqslant t} L(z)$$

with distribution function $V(x, t) = P(W(t) \leqslant x)$. The process $W(t)$ is obtained from $L(t)$ by using a retaining barrier at 0. Rewriting the presentation of $W(t)$ as $W(t) = max_{0 \leqslant z \leqslant t}[L(t) - L(z)]$ and comparing it with Eq. (22), it follows that $W(t)$ and $M(t)$ have the same distribution, see Seal (1972), [14]. Therefore,

(25) $$1 - \Psi(u, t) = V(u, t).$$

Let $V(u) = \lim_{t \to \infty} V(t, u)$ be the stationary distribution of $W(t)$. Therefore

(26) $$1 - \Psi(u) = V(u).$$

The main idea in computing the distribution $V(u)$ is as follows: let for a particular value of u, $D(u,t)$ be the total amount of time the process $W(t)$ is below a predetermined level u before time t. Then, due to the Strong Law of Large Numbers, e.g., see Sen and Singer (1993), [15], it is true that

$$\lim_{t\to\infty} \frac{D(u,t)}{t} = V(u). \tag{27}$$

Based Eq. (26) and Eq. (27), the stationary distribution $V(u)$ can be computed by simulating the trajectories of the process $W(t)$. For more details on this approach, pictorial representation of the relationship between $W(t)$ and $L(t)$ and an efficient algorithm for computing $V(u)$, see Dufresne and Gerber (1989), [5].

5.2 Results

Next, we consider the case of exponentially distributed claims and no initial capital $u = 0$. We verify the correctness of our simulation code, which is base on the discussion in subsection 5.1, by comparing the results for the probability of ruin produced by the simulator, given in column "analytical" with the value of the probability of ruin for the same model parameters computed using Eq.(19), given in column "simulated", see Table 1 below:

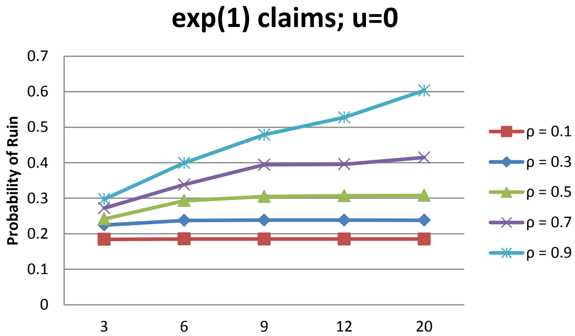
λ	k	ρ	analytical Exp(1)	simulated Exp(1)
2.0	10	0.4	0.256249	0.256224
3.0	6	0.2	0.287373	0.287782
1.5	4	0.8	0.256723	0.256906
1.0	15	0.6	0.207745	0.207708
2.5	3	0.9	0.344623	0.344557

Table 1

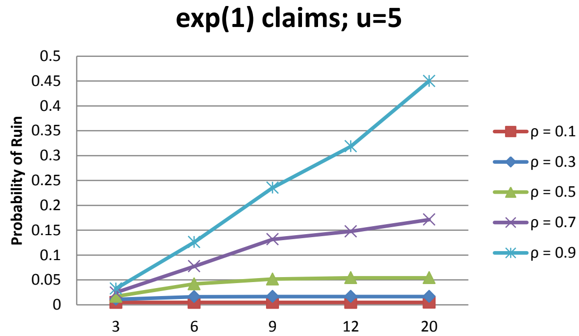
As expected the "analytical" and "simulated" results are very close. So, we use our simulator, written in MATHEMATICA, to compute a reasonable approximation of the probability of ruin for non-exponentially distributed claims and non-zero initial capital $u \neq 0$ and our findings are summarised below.

5.2.1 Case 1: Exponentially distributed claims

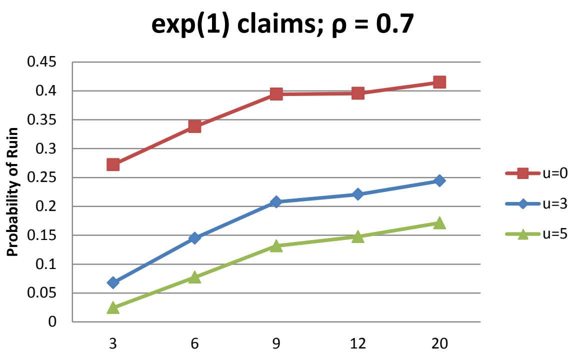
Here, we extend section 4.2 by presenting some simulation results for the case of exponentially distributed claims with non-zero initial capital.



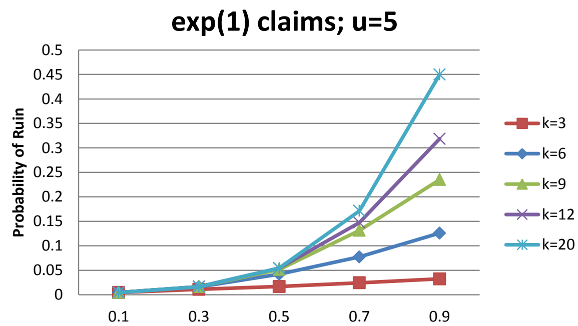
(a) $u = 0$, dependence on k and ρ



(b) $u = 5$, dependence on k and ρ



(c) $\rho = 0.7$, dependence on k and u



(d) $u = 5$, dependence on ρ and k

Figure 1: Probability of ruin: exponentially distributed claims

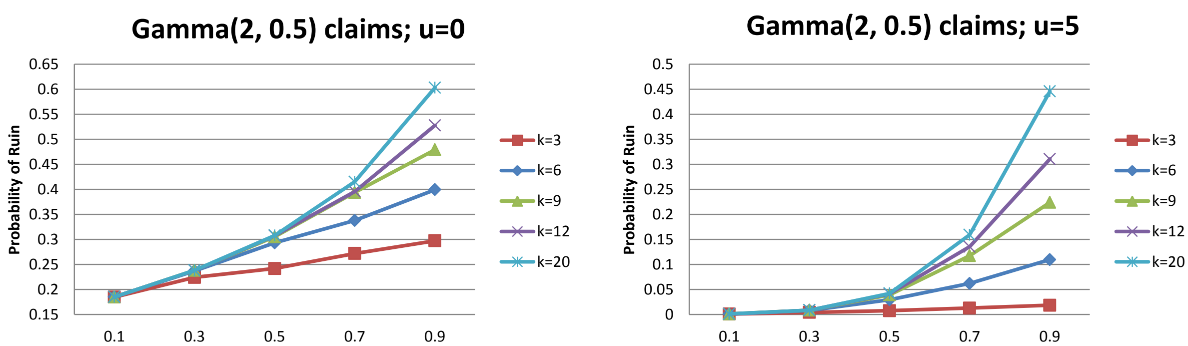
Comparing part(a) and part(b) of Figure 1, it is easy to see that the probability of ruin is shifted downwards as the initial capital increases. If the initial capital is $u = 0$, the smallest values for the probability of ruin is just above 0.18 for $\rho = 0.1$, whereas the analogous value for $u = 5$ is just below 0.005. The depicted overall dependence on ρ, regardless of the value of the initial capital, is as expected, the probability of ruin increases as ρ increases. The overall trends depicted in part(c) and part(d) of Figure 1 also agree with our intuition. Namely, for a fixed value of ρ, the probability of ruin is higher for low values of the initial capital and it increases on k. It is worth to point out the sharp increase of the probability of ruin for large values of ρ and large k, as shown in part(d) of Figure 1.

5.2.2 Case 2: Gamma distributed claims

Next, we consider gamma distributed claims with parameters α and β, i.e., the density function of the claim sizes is

$$f(x) = \frac{1}{\beta\gamma(\alpha)} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}}, \quad x \geqslant 0.$$

Suppose that $\alpha = 2$ and $\beta = 0.5$. In this case the mean values of the claims are $EZ_i = \alpha\beta = 1$. We present results for different values of the model parameters u, k and ρ.



(a) $u = 0$, dependence on ρ and k

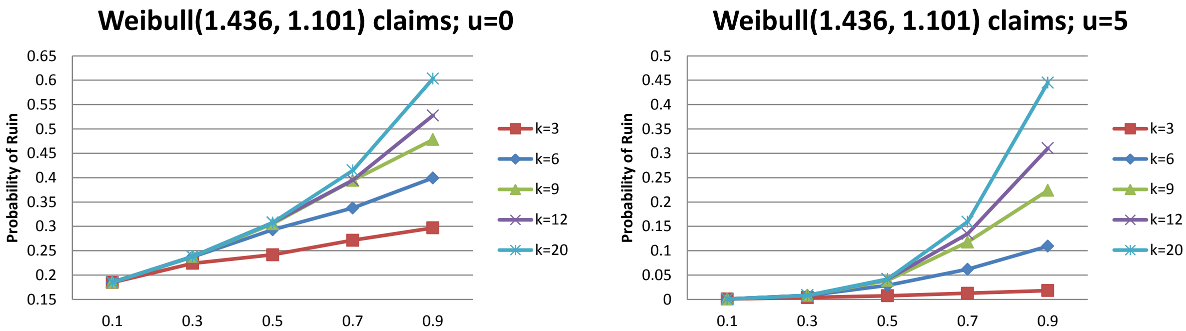
(b) $u = 5$, dependence on ρ and k

Figure 2: Probability of ruin: gamma distributed claims

The trends observed for the gamma distributed claims are similar to the one we have presented and discussed for the case of exponentially distributed claims in subsection 5.2.1. Here we depict the dependence of the probability of ruin from u, for similar ρ and k. Overall the probability of ruin for lower value of the capital u is higher, similar to what we have observed in the exponential case. In addition we see that for high values of u, and ρ, k have a strong impact on the probability of ruin, e.g., see for $\rho = 0.9$, and $u = 0$ the range of the probability of ruin is approximately $(0.3, 0.6)$, whereas for $u = 5$ this range is much larger, approximately $(0.1, 0.45)$.

5.2.3 Case 3: Weibull distributed claims

Next we focus on Weibull with parameters $\alpha = 1.43552259$ and $\beta = 1.1013206$ distributed claims. Here α is the shape parameter and β is the scale parameter. The parameters of the Weibull and gamma distributions were selected so that the three claim distributions considered in sections 5.2.1, 5.2.2 and 5.2.3 have the same expectation $\mu = 1$ and the Weibull and gamma claims have the same variance.



(a) $u = 0$, dependence on ρ and k

(b) $u = 5$, dependence on ρ and k

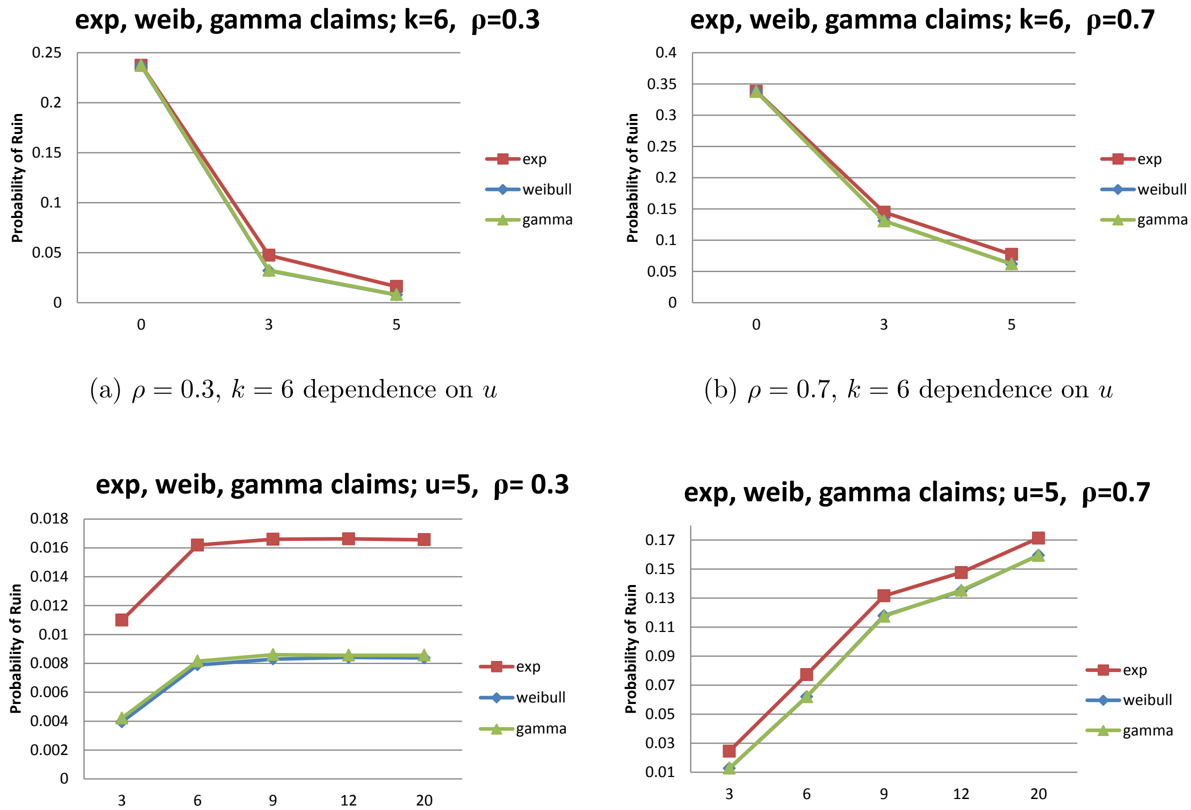
Figure 3: Probability of ruin: Weibull distributed claims

We were quite surprised to see that the behavior of the probability of ruin under Weibull distributed claims, part(a) and part(b) in Figure 3, mimics quite closely the behavior of this probability for gamma distributed claims. So, then the natural question is: under a risk model based on the Pólya - Aeppli process of order k, are the mean value and the variance of the claim distribution what determines the probability of ruin, i.e., the actual form of the claim distribution does not affect the probability of ruin. At this point of our study we are not able to answer to this question and further experimental and theoretical work is needed to address it.

5.3 Sensitivity analysis regarding the claim distribution

Here we look at the impact of the claim distribution on the probability of ruin and provide an illustration of our findings.

In part(a) and part(b) in Figure 4, we fix the value of the parameter $k = 6$, and illustrate the dependence of the probability of ruin on u for two different values of ρ. As expected the probability of ruin is a decreasing function of the initial capital u and its value is shifted upwards for increasing values of parameter ρ. In addition, the values of the probability of ruin are not very different for the three chosen claim distribution, with slightly higher values of this probability for exponential claims than for the remaining two distributions. In parts (c) and (d) in Figure 4, we fix $u = 5$ and depict the dependence of the probability of ruin on k for two different values of ρ. It is easy to see that the probability of ruin behavior is as expected, but the interesting observation here is that the exponential case provides an upper bound for the probability of ruin, which we have also observed in all of our numerical



(a) $\rho = 0.3$, $k = 6$ dependence on u

(b) $\rho = 0.7$, $k = 6$ dependence on u

(c) $u = 5$, $\rho = 0.3$ dependence on k

(d) $u = 5$, $\rho = 0.7$ dependence on k

Figure 4: Probability of ruin: sensitivity on the claim distribution

experiments. Therefore another interesting question could be posted: is this observation due to our choice of the model parameters or it is a generally true statement for the risk model based on the Pólya - Aeppli process of order k. Also it is interesting to address the following: are there any condition on the mean and the variance of the claim size distributions that will guaranty the satisfaction of some specified inequalities on the related ruin probabilities. Again further numerical and theoretical studies are needed to gain some insight on these interesting questions.

6 Conclusions

In the present study we have defined the Pólya - Aeppli process of order k as a compound Poisson process with truncated geometric compounding distribution with success probability $1 - \rho > 0$. We have illustrated that the Pólya - Aeppli process of order k can be represented as a sum of k independent Poisson processes and discussed some possible application of this process in risk theory. We have studies the probability of ruin for the related risk model, called a Pólya - Aeppli of order k risk model, and have derived an exact expression for the ruin probability in the particular case of zero initial capital. Also, we summarised and adopted a simulation approach, given in Dufresne and Gerber (1989), [5] for our particular model. Using this simulation approach we provide results for more general cases of the model, such as non-exponential claim distribution and non-zero initial capital. The simulation results open for discussion several very interesting questions related to the probability of ruin for Pólya - Aeppli of order k risk model. These questions, see subsections 5.2.3 and 5.3 will be addressed in our future work. Another interesting extension of this study would be to develop a multivariate version of the proposed model, along the lines summarized in Anastasiadis and Chukova (2012), [1].

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References

[1] Anastasiadis S., Chukova S. (2012). Multivariate Insurance Models: An Overview, *Insurance: Mathematics and Economics* **51**, 222-227 6

[2] Aki S., Kuboku H., and Hirano K. (1984). On discrete distributions of order k. *Annals of the Institute of Statistical Mathematics* **36**, 431 - 440. 2

[3] Balakrishnan, N., Koutras, M. V. (2002). *Runs and Scans with Applications*. Wiley Series in Probability and Statistics. New York: John Wiley & Sons. 2

[4] Charalambides Ch.A. (1986). On Discrete Distributions of order k, *Ann. Inst. Statist. Math.* **38**, 557 - 568. 2

[5] Dufresne F. and Gerber H. U. (1989). Three methods to calculate the probability of ruin, *ASTIN Bull.,* **19(1)**, 71 - 90. 5, 5.1, 5.1, 6

[6] Grandell J. (1991). *Aspects of Risk Theory*, Springer - Verlag, New York. 1

[7] Johnson N. L., Kemp A. W. and Kotz S. (2005). *Univariate Discrete Distributions*, Third edition, John Wiley & Sons, Hoboken, New Jersey. 2

[8] Minkova L.D. (2004). The Pólya - Aeppli process and ruin problems, J. Appl. Math. Stoch. Analysis, 3, 221-234. 3.1

[9] Minkova L.D. (2010). The Pólya - Aeppli distribution of order K, Comm. Statist. Theory and Methods, 39, 408 - 415. 3

[10] Patil G. P. (1962). Certain properties of the generalized power series distribution. *Ann. Inst. Statist. Math.,* **14**, 179 - 182. 2

[11] Philippou A. N., Georgiou C. and Philippou G. N. (1983). A generalized geometric distribution and some of its properties, *Statistics & Probability Letters,* **1**, 171 - 175. 2

[12] Philippou A. N. and Makri F. S. (1986). Successes, runs and longest runs, *Statistics & Probability Letters,* **4**, 211 - 215. 2

[13] Rolski T., Schmidli H., Schmidt V. and Teugels J. (1999). *Stochastic Processes for Insurance and Finance,* John Wiley & Sons, Chichester. 1

[14] Seal H. L. (1972) Risk Theory and the Single Server Queue, Journal of the Swiss Association of Actuaries, 171 - 178. 5.1

[15] Sen, P. K and Singer, J. M. (1993). *Large sample methods in statistics*, Chapman & Hall, Inc. 5.1

[16] Willmot G.E. and Lin X.S. (2001) *Lundberg Approximations for Compound Distributions with Insurance Applications*, Springer. 4.2