KMS States on Groupoid C*-Algebras

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2 November 2020

Introduction

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- groupoids and their C*-algebras

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- Neshveyev gives a formula for the KMS states on groupoid C*-algebras

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1943	Gelfand and Naimark characterised C*-algebras	

History of C*-algebras

Since then, the subject of C*-algebras has evolved into a huge mathematical endeavour interacting with several areas of mathematics and theoretical physics.

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(v) $\|a^* a\| = \|a\|^2$. — A is a C*-algebra

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- $C(X)$ for a compact Hausdorff space X.

$$(f_1 f_2)(x) = f_1(x) f_2(x), \quad f^*(x) = \overline{f(x)}, \quad \|f\| = \sup_{x \in X} |f(x)|.$$

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- $B(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$.

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \|T\| = \sup_{\|z\| \leq 1} \|Tz\|.$$

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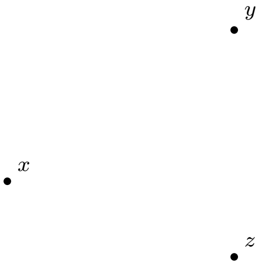
We can understand a groupoid as a set of arrows connecting points in the space.

Groupoids

Idea



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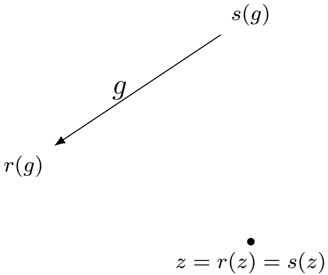

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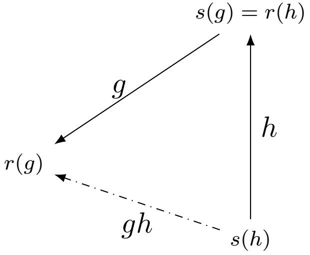

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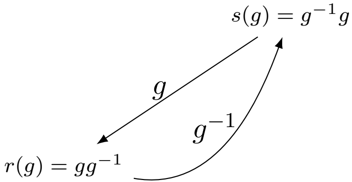

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- $g \mapsto g^{-1}$

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Examples

Let $G = \mathbb{R}^2 \times \mathsf{GL}_2(\mathbb{R})$, Let $G = \mathbb{R}^2 \times \mathsf{GL}_2(\mathbb{R})$, and $G^{(0)} = \{(x, I) : x \in \mathbb{R}^2\}$.

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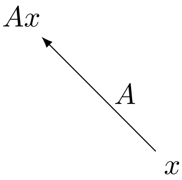

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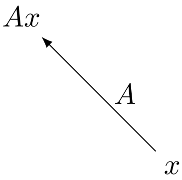

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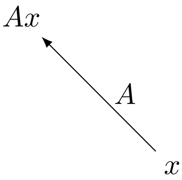

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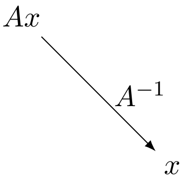

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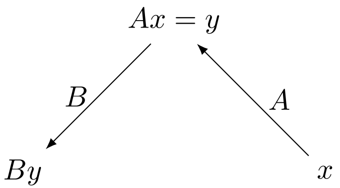

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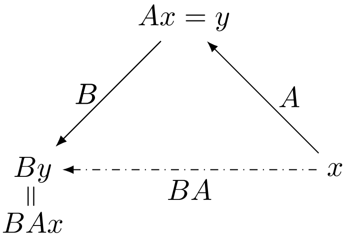

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- $(y, B)(x, A) = (x, BA)$

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The matrix operations induce the following operations:

$$f^*(x, y) = \overline{f(y, x)}, \quad (f_1 \cdot f_2)(x, y) = \sum_{z \sim x} f_1(x, z) f_2(z, y).$$

Groupoids

Functions on the groupoid

Analogously, we will define the following operations on $C_c(G)$, for a groupoid G:

$$f^*(g) = \overline{f(g^{-1})}, \qquad (f_1 \cdot f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

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Before doing that, we need to define a topology on G and study its properties.

Topological groupoids

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Definition

A topological groupoid is *étale* if the maps r and s are local homeomorphisms.

Here we will assume that every groupoid G is locally compact Hausdorff second countable étale.

Groupoid C*-algebras

Operations on $C_c(G)$

We equip the space

$$C_c(G) = \{f : G \to \mathbb{C} \text{ st } f \text{ is continuous with compact support}\}$$

with the operations

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A $*$-*representation* of $C_c(G)$ is a linear map $\pi : C_c(G) \to B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and the following properties hold:

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Theorem

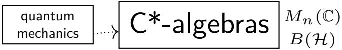
There exists a C-algebra $C^*(G)$ such that $C_c(G)$ is dense in $C^*(G)$ and*

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ is a } *\text{-representation of } C_c(G)\},$$

for all $f \in C_c(G)$.

Many classes of C*-algebras can be described by groupoid C*-algebras. For example, AF algebras and graph algebras.

Outline



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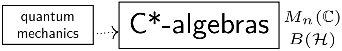

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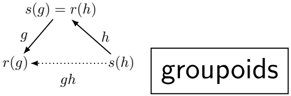

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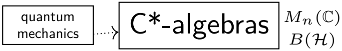

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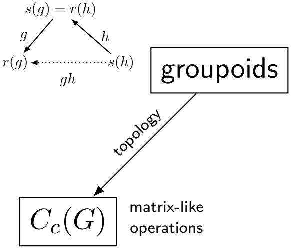

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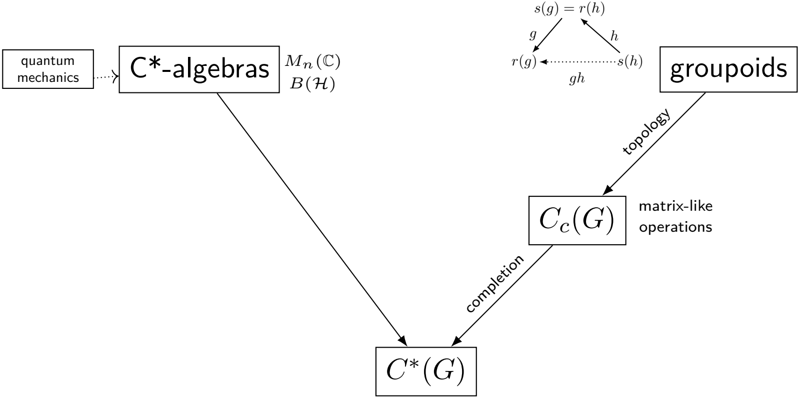

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Aplication: KMS states

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KMS states describe equilibrium states in quantum statistical mechanics.

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KMS states

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Example

Let $H \in M_n(\mathbb{C})$ be self-adjoint. Define $\tau_t : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

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$$\varphi(a\tau_{i\beta}(b)) = \varphi(ba) \quad \text{for } a, b \in A_0.$$

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Then $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ given by

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In fact,

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A KMS$_\beta$-state must satisfy

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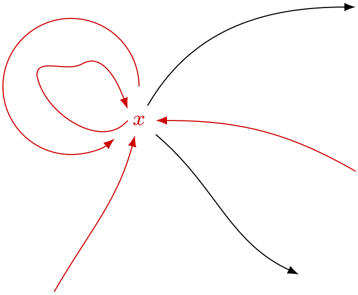
Now we study Theorem 1.3 of [4], by Neshveyev, which describes all KMS states φ on $C^*(G)$ by the formula

$$\varphi(f) = \int_{G^{(0)}} \sum_{g \in G_x^x} f(g)\varphi_x(u_g)d\mu(x), \quad f \in C_c(G).$$

Moreover, it gives a one-to-one correspondence between the KMS states and pairs $(\mu, \{\varphi_x\}_{x \in G^{(0)}})$ satisfying certain conditions.

Neshveyev's theorem

Some notation



$$G^x = r^{-1}(\{x\}).$$

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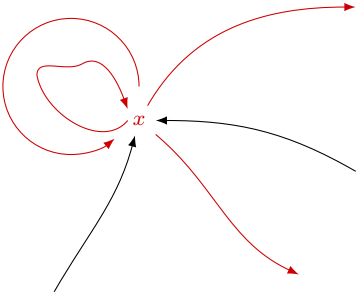

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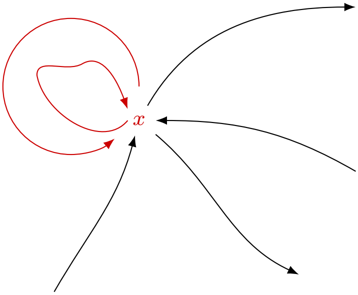

$$G_x^x = G_x \cap G^x.$$

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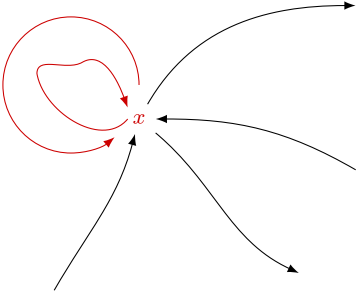

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Note that G_x^x is a group with identity x.

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- $\{\varphi_x\}_{x \in G^{(0)}}$ satisfies a few more conditions

Neshveyev's theorem

Theorem

[4, Theorem 1.3] There exists a one-to-one correspondence between KMS$_\beta$-states on $C^*(G)$ and pairs $(\mu, \{\varphi_x\}_{x\in G^{(0)}})$ consisting of a probability measure μ on $G^{(0)}$ and a μ-measurable field of states φ_x on $C^*(G_x^x)$ such that:

(i) *μ is quasi-invariant with Radon-Nikodym derivative $e^{-\beta c}$;*

(ii) *$\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ for every $g \in G_x^x$ and $h \in G_x$, for μ-a.e. x; in particular, φ_x is tracial for μ-a.e. x;*

(iii) *$\varphi_x(u_g) = 0$ for all $g \in G_x^x \setminus c^{-1}(0)$, for μ-a.e. x.*

Neshveyev's theorem

Open bisections

In order to show that φ given by the formula in the previous slide satisfies the KMS condition:

$$\varphi(f_1 \cdot \tau_{i\beta}(f_2)) = \varphi(f_2 \cdot f_1) \quad \text{for } f_1, f_2 \in C_c(G),$$

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he uses the following property of locally compact Hausdorff second countable étale groupoids:

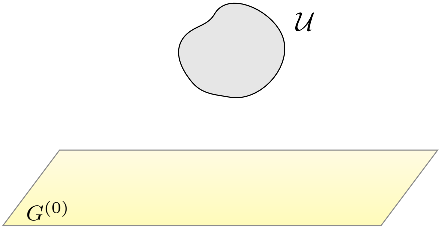
- A function $f \in C_c(G)$ can be written as a finite sum $f = f_1 + \cdots + f_n$. Each $f_i \in C_c(\mathcal{U}_i) \subset C_c(G)$ has support in an *open bisection* $\mathcal{U}_i$.

Neshveyev's theorem

Open bisections

If $\mathcal{U} \subset G$ is an open bisection, $\mathcal{U}$ is open and

$$r : \mathcal{U} \to r(\mathcal{U}), s : \mathcal{U} \to s(\mathcal{U}) \text{ are homeomorphisms.}$$



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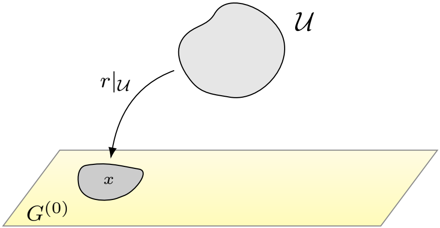

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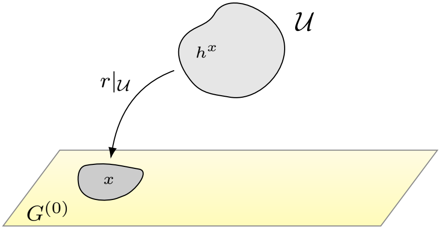

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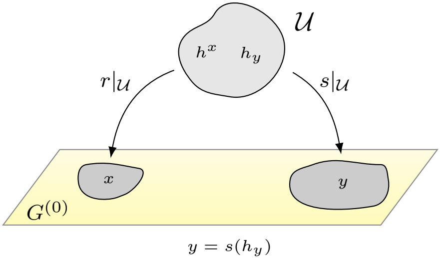

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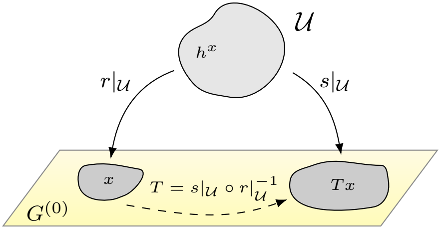

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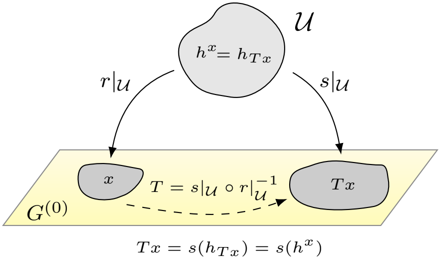

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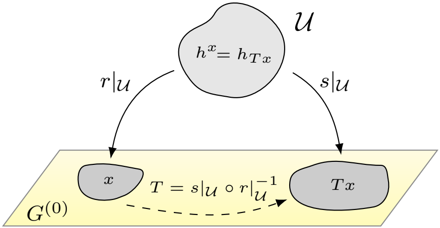

$Tx = s(h_{Tx}) = s(h^x)$

If $f_2 \in C_c(\mathcal{U})$, we can find easier formulas for $\varphi(f_1 \cdot \tau_{i\beta}(f_2))$ and $\varphi(f_2 \cdot f_1)$.

Conclusion

Using topological properties of groupoids, we can study some properties of groupoid C*-algebras in more detail.

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