The distribution of zeros of the derivative of the Riemann Zeta function via random unitary matrices

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Theorem (Speiser 1935, *Math. Ann.* 110, 514)

Riemann's hypothesis is equivalent to the non-vanishing of $\zeta'(s)$ *in the strip* $s < \frac{1}{2}$.

E. Dueñez et al 2010 Nonlinearity **23**, 2599

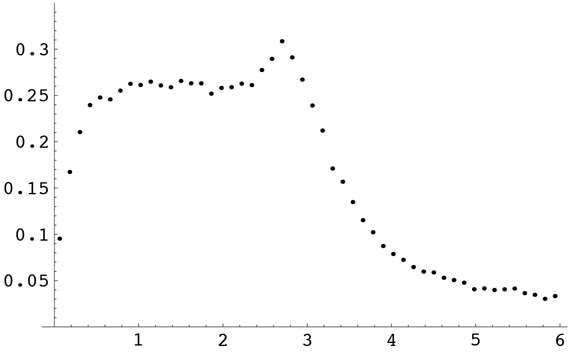

Figure: Normalized distribution of the real part of the zeros of $\zeta'(s)$. Data is for the approximately 10^5 zeros with imaginary part in $[10^6, 10^6 + 60000]$.

Adolf Hurwitz 1897, *Über die Erzeugung der Invarianten durch Integration*

Ex. $N = 2$ Case: Euler angles $0 \le \alpha < 2\pi$, $0 \le \psi < 2\pi$, $0 \le \phi \le \frac{\pi}{2}$

1. $$U = e^{i\alpha_0} \begin{pmatrix} e^{i\alpha} \cos\phi & e^{i\psi} \sin\phi \\ -e^{-i\psi} \sin\phi & e^{-i\alpha} \cos\phi \end{pmatrix}$$

2. $$\begin{aligned} U^\dagger dU = {} & i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} d\alpha_0 + i\cos\phi \begin{pmatrix} \cos\phi & e^{i(\psi-\alpha)} \sin\phi \\ e^{i(\alpha-\psi)} \sin\phi & -\cos\phi \end{pmatrix} d\alpha \\ & + i\sin\phi \begin{pmatrix} \sin\phi & e^{i(\psi-\alpha)} \cos\phi \\ e^{i(\alpha-\psi)} \cos\phi & \sin\phi \end{pmatrix} d\psi + \begin{pmatrix} 0 & e^{i(\psi-\alpha)} \\ -e^{i(\alpha-\psi)} & 0 \end{pmatrix} d\phi \end{aligned}$$

3. $$\mathrm{Tr}\left(dU dU^\dagger\right) = 2(d\alpha_0)^2 + 2\cos^2\phi (d\alpha)^2 + 2\sin^2\phi (d\psi)^2 + 2(d\phi)^2$$

4. $$\sqrt{\det(g)} = 4\cos\phi \sin\phi$$

5. $$d\Omega = 4\cos\phi \sin\phi d\alpha_0 d\alpha d\psi d\phi$$

Hermann Weyl 1946, *The classical groups: Their invariants and representations*

1. Eigenvalues

$$\zeta_{1,2} = e^{i\theta_{1,2}} = e^{i\alpha_0}\left[\cos\phi\cos\alpha \pm iD\right], \quad D^2 = 1 - \cos^2\phi\cos^2\alpha$$

2. Eigenvectors

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \quad |u_j|^2 + |v_j|^2 = 1$$

$$|u_{1,2}|^2 = \frac{1}{2}\left[1 \pm \frac{\cos\phi\sin\alpha}{D}\right], \quad |v_{1,2}|^2 = \frac{1}{2}\left[1 \mp \frac{\cos\phi\sin\alpha}{D}\right]$$

3. Jacobian for $\{\alpha_0, \alpha, \phi, \psi\} \mapsto \{\theta_1, \theta_2, u_1, u_2\}$

$$J = \frac{e^{2i\alpha_0}\cos\phi\sin\phi}{D^2}, \quad D^2 = \frac{1}{4}|e^{i\theta_1} - e^{i\theta_2}|^2$$

4. Volume

$$4\cos\phi\sin\phi d\alpha_0 d\alpha d\psi d\phi = |e^{i\theta_1} - e^{i\theta_2}|^2 d\theta_1 d\theta_2 d|u_1|^2$$

Class function $f(\mathrm{Tr}U, \mathrm{Tr}U^2, \mathrm{Tr}U^3, \ldots)$

$$\mathbb{E}_{U(N)}[f] = \frac{1}{(2\pi)^N N!} \int_{-\pi}^{\pi} d\theta_1 \ldots \int_{-\pi}^{\pi} d\theta_N f(\theta_1, \ldots, \theta_N) \prod_{1\le j<k\le N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^2$$

$U(N)$ characteristic polynomial, $\zeta_j = e^{i\theta_j}$

$$\Lambda(z, U) = \det(z - U) = \prod_{j=1}^{N} (z - \zeta_j)$$

Model for the Riemann zeta function $\zeta(1/2 + it)$ is

$$\mathbb{E}_{U(N)}[\Lambda(z, U)]$$

Identification of the rank, $N = \frac{1}{2\pi} \log t$

Statistic	RMT prediction	Reference
Pair Correlation	$S(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$	H. L. Montgomery, 1973 A. Odlyzko, 1987
Distribution of Spacings between the Zeros $s_n = \frac{1}{2} + it_n$	$S_{nn}(t) = -\frac{\sigma_1(2\pi\rho t)}{t} \exp \int_0^{\pi\rho t} \frac{\sigma_1(2u)}{u} du$ σ_1 function of Painlevé V	P. Forrester & A. Odlyzko, 1996
Moments of $\zeta(\frac{1}{2} + it)$	$\mathbb{E}_{U(N)} \lvert \det(U - e^{i\theta} I) \rvert^{2k} = \prod_{j=0}^{N-1} \frac{j!(j+2k)!}{((j+k)!)^2}$	Keating & Snaith, 2000 B. Conrey et al, 2008

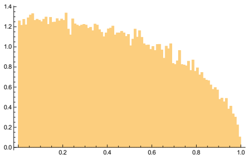

Figure: Sample size 5×10^4

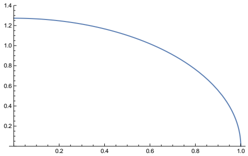

Figure: $\rho_2(r) = \dfrac{4}{\pi}\sqrt{1 - r^2}$

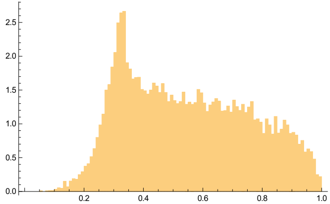

Figure: $N = 3$. Sample size 10^4

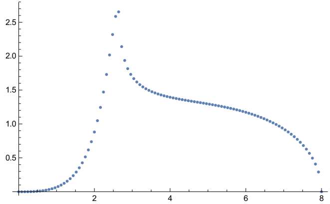

Figure: PTO for equation

$$\rho_3(r) = \frac{54\sqrt{3}}{\pi^2} r^3 \sqrt{1-r^2} \int_{\theta_0}^{\pi} d\theta \frac{N_0 N_1 R}{D^3}$$

where

$$\theta_0 = \begin{cases} 0, & r < \frac{1}{3} \\ \cos^{-1}\left(\frac{3r^2+1}{4r}\right), & r > \frac{1}{3} \end{cases}$$

and

$$\begin{aligned} D &= 1 - 8r^2 + 25r^4 + 36r^6 - 12(7r^2-1)r^3\cos(\theta) + 6(2r-1)(2r+1)r^2\cos(2\theta), \\ R^2 &= 1 - 7r^2 + 27r^4 + 27r^6 - 8(3r-1)(3r+1)r^3\cos(\theta) + 8(3r^2-1)r^2\cos(2\theta) \\ N_0 &= 1 - 3r^2 + 13r^4 + 9r^6 - 30r^5\cos\theta - 6(1-3r^2)r^2\cos(2\theta) - 2(2r^2-1)r\cos(3\theta) \\ N_1 &= \left(1 - 2r\cos(\theta) + r^2\right)^2 \left(1 + 2r\cos(\theta) - 3r^2\right)^2 \end{aligned}$$

$N = 6, 10$

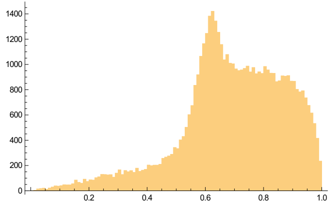

Figure: $N = 6$. Sample size 10^4

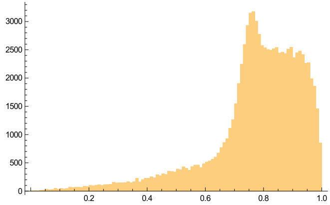

Figure: $N = 10$. Sample size 10^4

Mezzadri 2003, Dueñez et al 2010
Let ζ be a root of $\Lambda'(z)$ and define the random variable

$$s := N(1-r), \quad r = |\zeta|$$

Denote by $Q(s;N)$ the probability density function of s.

- Mezzadri showed that the limit

$$Q(s) := \lim_{N\to\infty} Q(s;N)$$

  exists, and proved that

$$Q(s;N) \sim \frac{1}{s^2}, \quad N \to \infty, \quad s \to \infty$$

  with $s = o(N)$.
- Dueñez et al proved that

$$Q(s) \sim \frac{4}{3\pi} s^{1/2}, \quad s \to 0$$

Single-particle and two-particle Correlation functions are defined

$$\rho_N^{(1)}(\theta_1) = \frac{N}{Z_N} \int_{\mathbb{T}} \frac{d\zeta_2}{2\pi i \zeta_2} \cdots \int_{\mathbb{T}} \frac{d\zeta_N}{2\pi i \zeta_N} \prod_{j=1}^{N} w(\zeta_j) \prod_{1 \le j < k \le N} |\zeta_j - \zeta_k|^2$$

$$\rho_N^{(2)}(\theta_1, \theta_2) = \frac{N(N-1)}{Z_N} \int_{\mathbb{T}} \frac{d\zeta_3}{2\pi i \zeta_3} \cdots \int_{\mathbb{T}} \frac{d\zeta_N}{2\pi i \zeta_N} \prod_{j=1}^{N} w(\zeta_j) \prod_{1 \le j < k \le N} |\zeta_j - \zeta_k|^2$$

Normalisation

$$Z_N = \int_{\mathbb{T}} \frac{d\zeta_1}{2\pi i \zeta_1} \cdots \int_{\mathbb{T}} \frac{d\zeta_N}{2\pi i \zeta_N} \prod_{j=1}^{N} w(\zeta_j) \prod_{1 \le j < k \le N} |\zeta_j - \zeta_k|^2$$

Determinantal structure

$$\rho_N^{(1)}(\theta_1) = w(\zeta_1) K_N(\zeta_1, \bar{\zeta}_1)$$

$$\rho_N^{(2)}(\theta_1, \theta_2) = w(\zeta_1) w(\zeta_2) \left[K_N(\zeta_1, \bar{\zeta}_1) K_N(\zeta_2, \bar{\zeta}_2) - K_N(\zeta_1, \bar{\zeta}_2) K_N(\zeta_2, \bar{\zeta}_1) \right]$$

Christoffel-Darboux kernel $K_N(\zeta_1, \bar{\zeta}_2)$

Let $\rho_N^{(1)}(\zeta_1)$, $\rho_N^{(2)}(\zeta_1, \zeta_2)$ be the correlations corresponding to the weight $w(\zeta)$ with normalisation Z_N and $f(\zeta), g(\zeta)$ be analytic functions.

Then

$$\mathbb{E}_{U(N)}\left[\sum_{j=1}^{N} f(\zeta_j) \times \sum_{k=1}^{N} g(\bar{\zeta}_k) \times \prod_{l=1}^{N} w(\zeta_l)\right]$$
$$= \frac{Z_N}{N!}\left\{\int_{\mathbb{T}} \frac{d\zeta_1}{2\pi i\zeta_1} f(\zeta_1)g(\bar{\zeta}_1)\rho_N^{(1)}(\zeta_1) + \int_{\mathbb{T}} \frac{d\zeta_1}{2\pi i\zeta_1} \int_{\mathbb{T}} \frac{d\zeta_2}{2\pi i\zeta_2} f(\zeta_1)g(\bar{\zeta}_2)\rho_N^{(2)}(\zeta_1, \zeta_2)\right\}$$

Define a system of bi-orthogonal polynomials $\{\varphi_n(z), \bar{\varphi}_n(z)\}_{n=0}^{\infty}$
with respect to the weight $w(z)$ on the unit circle by the orthogonality relation

$$\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\varphi_m(\zeta)\bar{\varphi}_n(\bar{\zeta}) = \delta_{m,n}.$$

Reciprocal polynomial defined by $\varphi_n^*(z) := z^n \bar{\varphi}_n(1/z)$
The summation identity gives Christoffel-Darboux kerenl

$$K_n(z,\bar{\zeta}) = \sum_{j=0}^{n} \varphi_j(z)\bar{\varphi}_j(\bar{\zeta}) = \frac{\varphi_n^*(z)\overline{\varphi_n^*}(\bar{\zeta}) - z\bar{\zeta}\varphi_n(z)\bar{\varphi}_n(\bar{\zeta})}{1 - z\bar{\zeta}}$$

holds for $z\bar{\zeta} \neq 1$ and $n \geq 0$.
Two non-polynomial solutions $\epsilon_n(z), \epsilon_n^*(z)$ to the recurrence relations

$$\epsilon_n(z) = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} \frac{\zeta + z}{\zeta - z} w(\zeta)\varphi_n(\zeta)$$
$$\epsilon_n^*(z) = \frac{1}{\kappa_n} - \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} \frac{\zeta + z}{\zeta - z} w(\zeta)\varphi_n^*(\zeta)$$

for $n \geq 1$.

Let λ_j, $j = 1, \dots, N-1$ denote the zeros of $\Lambda'(z)$

1. Seek the average
$$\rho_N(z,\bar{z}) = \mathbb{E}_{U(N)}\Big[\sum_{j=1}^{N-1} \delta(z-\lambda_j)\delta(\bar{z}-\bar{\lambda}_j)\Big]$$

2. Distribution identity
$$\sum_{j=1}^{N-1} \delta(z-\lambda_j)\delta(\bar{z}-\bar{\lambda}_j) = \delta\Big(\frac{\Lambda'}{\Lambda}\Big)\delta\Big(\frac{\bar{\Lambda}'}{\bar{\Lambda}}\Big)\left|\frac{d}{dz}\frac{\Lambda'}{\Lambda}\right|^2$$

with
$$\frac{d}{dz}\frac{\Lambda'}{\Lambda} = -\sum_{j=1}^{N} \frac{1}{(z-\zeta_j)^2}$$

3. Thus
$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} d^2v\ \mathbb{E}_{U(N)}\Big[\sum_{j=1}^{N}\frac{1}{(z-\zeta_j)^2}\sum_{j=1}^{N}\frac{1}{(\bar{z}-\bar{\zeta}_j)^2}\exp\left[\frac{i}{2}\left(\bar{v}\sum_j\frac{1}{(z-\zeta_j)} + v\sum_j\frac{1}{(\bar{z}-\bar{\zeta}_j)}\right)\right]\Big]$$

1. Employing a linear-fractional transformation $\zeta = z + \dfrac{1 - z\bar{z}}{\bar{z} + t}$ we generate a semi-classical weight on the unit circle

$$w(t) = \frac{1}{(\bar{z} + t)^N (z + t^{-1})^N} \exp\left(-\frac{i}{2}\frac{1}{1 - z\bar{z}}(\bar{v}t + vt^{-1})\right)$$

2. This identifies our problem as one of the two-variable extensions of the Painlevé Equations

3. Substitute the Christoffel-Darboux formula.
Now have three classes of terms in the $U(N)$ average.
One example is

$$\int_{\mathbb{T}} \frac{dt_1}{2\pi i t_1} w(t_1)(\bar{z} + t_1)^2 \int_{\mathbb{T}} \frac{dt_2}{2\pi i t_2} w(t_2)(z + t_2^{-1})^2$$
$$\times \frac{t_2}{t_1 - t_2}\left[t_1^N \bar{\varphi}_N(t_1^{-1}) t_2^{-N} \varphi_N(t_2) - \varphi_N(t_1)\bar{\varphi}_N(t_2^{-1})\right]$$
$$\times \frac{t_1}{t_2 - t_1}\left[t_2^N \bar{\varphi}_N(t_2^{-1}) t_1^{-N} \varphi_N(t_1) - \varphi_N(t_2)\bar{\varphi}_N(t_1^{-1})\right]$$

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