Uniform spanning tree and loop erased random walk

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Loop erased random walk

Introduced by Greg Lawler in his PhD thesis (1980).

Let (S_n) be the SRW on $\mathbb{Z}^d$, started at 0. Let $A \subset \mathbb{Z}^d$.

The LERW of S run until it hits A is obtained by taking the path $(S_0, S_1, \ldots, S_{T_A})$ and then successively (chronologically) erasing loops until one is left with a self-avoiding path from S_0 to A. Call this $\mathrm{LEW}(S_0, A)$.

(Although the LERW gives paths which are self-avoiding walks, it is known that it is a different model than the SAW.)

Markov type property for LEW

The LEW is not Markov. However, it does have a Markov-type property.

Let $D \subset \mathbb{Z}^d$, and $L_0, \ldots, L_T$ be $\text{LEW}(x, D^c)$. Let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_m)$ be a finite self-avoiding path inside D started at x.

Suppose we want the law of L conditional on $\{(L_0, \ldots L_m) = \gamma\}$. Then this is the law of the loop erasure of a new random walk Y with $Y_0 = \gamma_m$, run until it leaves D. The process Y is the SRW S conditioned to exit D before it returns to hit the path γ

The conditioning of Y close to its starting point γ_m is very strong, so it can be technically challenging to use this Markov property in practice.

Length of LEW

Let $A_n = \mathbb{Z}^d - B(0,n)$, $L^{(n)} = \text{LEW}(0, A_n)$, and $M_n = |L^{(n)}|$, i.e. the length of the path $L^{(n)}$.

For $d \geq 4$ Lawler proved that $\mathbb{E}M_n \approx n^2$, i.e.

$$\lim_{n\to\infty} \frac{\log \mathbb{E}M_n}{\log n} \to 2.$$

For $d = 2$ Kenyon (2000) proved that $\mathbb{E}M_n \approx n^{5/4}$.

Theorem. (Lawler, 2014, Lawler and Viklund 2016). We have $\mathbb{E}M_n \sim cn^{5/4}$; in particular there exist constants c_i such that

$$c_1 n^{5/4} \leq \mathbb{E}M_n \leq c_2 n^{5/4}.$$

Set $\kappa = 5/4$; this is the growth exponent of the LEW in $d = 2$.

For $d = 3$ the growth exponent is not known, but has been proved to exist by Shiraishi.

Tails of M_n in $d = 2$

Question. What are the tails of M_n, i.e. the behaviour for large λ of

$$\mathbb{P}(n^{-\kappa} M_n \geq \lambda), \quad \mathbb{P}(n^{-\kappa} M_n \leq \lambda^{-1})?$$

Aside. We can ask the same behaviour about the SRW, ie the tails of $T_n = \min\{k \geq 0 : |S_k| > n\}$. In this case the answer is well known, and we have

$$e^{-c_1 \lambda} \leq \mathbb{P}(n^{-2} T_n \geq \lambda) \leq e^{-c_2 \lambda},$$

with a similar bound for the small tail.

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Theorem. (MB, Masson, 2010). Let $d = 2$. For all large n,

$$\mathbb{P}(n^{-\kappa} M_n \geq \lambda) \leq c e^{-c' \lambda},$$

$$\mathbb{P}(n^{-\kappa} M_n \leq \lambda^{-1}) \leq c e^{-c' \lambda^{4/5 - \varepsilon}}.$$

Can we get lower bounds? How sharp is the power $4/5 - \varepsilon$?

Uniform spanning tree (UST)

On a finite graph the UST is a spanning tree (i.e. a connected subgraph which is a tree and contains all the vertices) chosen uniformly at random.

Pemantle (1991) defined 'UST' on $\mathbb{Z}^d$ as limit of UST on cubes $[-N,N]^d$. (One gets a forest if $d \geq 5$).

Haggstrom (1995): UST is a limit as $q \to 0$ of the FK(p,q) random cluster model.

Wilson (1996) (following Aldous and Broder): algorithm for construction of UST from loop erased random walk (LEW).

Wilson's algorithm (1996)

Wilson's algorithm (Stated here for $\mathbb{Z}^d$; however it holds for any graph.)

(0) Choose (z_k) so that $\mathbb{Z}^d = \{z_0, z_1, \dots\}$.

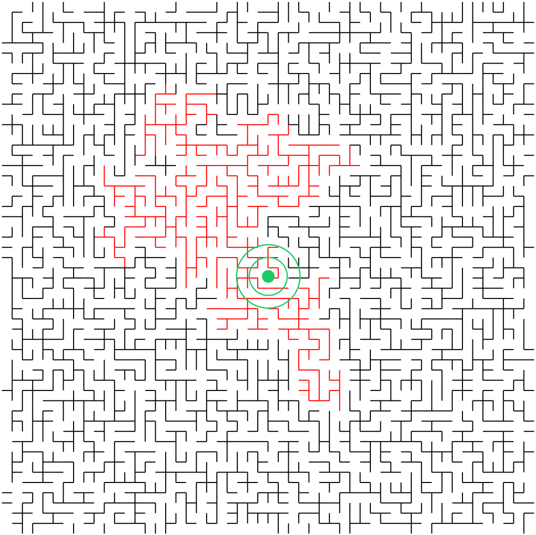
(1) Let $\mathcal{T}_0 = \{z_0\}$.

(2) For $k \geq 1$ let $\mathcal{T}_k = \mathcal{T}_{k-1} \cup \mathrm{LEW}(z_k, \mathcal{T}_{k-1})$.

(3) $\mathcal{U} = \cup_k \mathcal{T}_k$ is the UST in $\mathbb{Z}^d$, and the law does not depend on the particular sequence (z_k).

(If $d \geq 5$ then this algorithm gives a forest, not a tree, since $\mathrm{LEW}(x, \mathcal{T}_k)$ might be an infinite path which does not hit $\mathcal{T}_k$.)

UST in $d = 2$



WA implies that the geodesic path in $\mathcal{U}$ between x and y in the UST has the same law as a LEW from x to y, that is

$$d_{\mathcal{U}}(x,y) =_{(d)} |\mathrm{LEW}(x,\{y\})|.$$

To see this, just note that we can begin our construction of $\mathcal{U}$ by taking $\mathcal{T}_1 = \mathrm{LEW}(x,\{y\})$.

Most work has used WA in the 'forward' direction, i.e. to deduce properties of the UST from those of LEW.

However, we can also use it 'backwards', i.e. to obtain properties of the LEW from those of the UST.

Theorem 1. (MB, Croydon, Kumagai 2019+)
(a) Set $M_x = |\mathrm{LEW}(0, \{x\})|$. For $\lambda \geq 1$

$$\exp(-c_1\lambda^4) \leq \mathbb{P}\Big(M_x \leq \frac{|x|^\kappa}{\lambda}\Big) \leq \exp(-c_2\lambda^4). \qquad (1)$$

(b) The same bounds hold for $M_n = |\mathrm{LEW}(0, B(0,n)^c)|$.

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Remarks. (1) Suppose that $|x| = n$. We do not have the a.s. inequality $|\mathrm{LEW}(0, \{x\})| \geq |\mathrm{LEW}(0, B(0,n)^c)|$. As far as I know even stochastic domination has not been proved.
(2) Nevertheless, with some work one can obtain bounds for M_n from those for M_x, and vice versa.
(3) The upper bound uses known methods. MB, Masson proved

$$\mathbb{P}(n^{-\kappa}M_n \leq \lambda^{-1}) \leq e^{-c'\lambda^{4/5-\varepsilon}}.$$

Going back to their proof, one finds one can improve it to $\exp(-c\lambda^4)$.
(4) The 4 here is actually $1/(\kappa-1)$; recall that $\kappa = 5/4$.

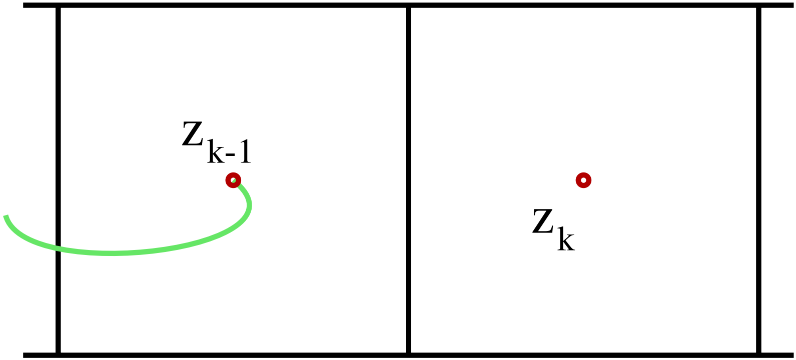
Sketch for lower bound.

For simplicity take $x = (x_1, 0)$ with $x_1 > 0$.
Let $N \geq 1$, and (WVMLG) assume $m = x_1/N \in \mathbb{N}$. Tile $\mathbb{Z}^2$ with boxes side m and centres in $m\mathbb{Z}^2$.
Let $z_0 = 0$, $z_j = (jm, 0)$ for $1 \leq j \leq N$ and write Q_j for the box side m centre z_j.

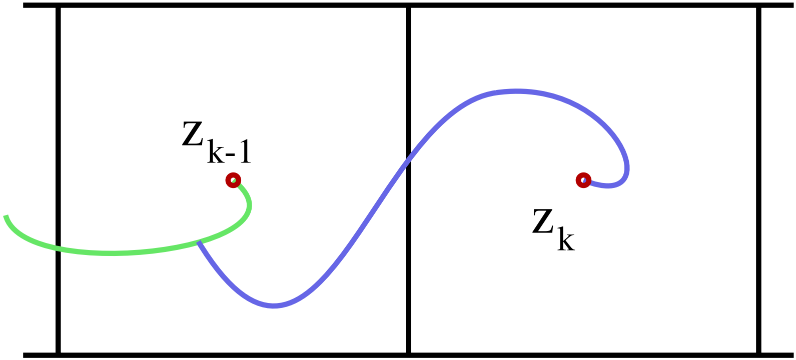
Run WA with the initial part of the sequence being $\{z_0, z_1, \dots, z_N\}$.
Write S^{z_k} for a SRW started at z_k and killed on its first hit on T_{k-1}, and set

$$L_k = LE(S^{z_k}), \quad T_k = T_{k-1} \cup L_k.$$

We declare stage k a *success* if S^{z_k} hits T_{k-1} before it leaves $Q_k \cup Q_{k-1}$, and the length of $\mathrm{LEW}(z_k, T_{k-1})$ is about m^κ.



z_{k-1}
z_k


z_{k-1}
z_k

Sketch for lower bound II

For $k \geq 1$ the probability of success, given that the previous stages have all been successful, is at least $p = e^{-a_1} > 0$. (Independent of m.) Let F be the event that all N stages are successful, so that

$$\mathbb{P}(F) \geq e^{-a_1 N}.$$

On F we have $d_{\mathcal{U}}(0, x) \approx Nm^\kappa$, while $|x| = Nm$. So

$$\frac{d_{\mathcal{U}}(0, x)}{|x|^\kappa} \approx \frac{Nm^\kappa}{N^\kappa m^\kappa} = \frac{1}{N^{\kappa-1}} = \frac{1}{N^{1/4}}.$$

Set $\lambda = N^{1/4}$ to obtain the lower bound.

THANK YOU

Length of LER is not stochastically monotone

Example (Angel). Look at $\mathbb{Z}$, ands let

$$D_1 = \{-1, 0, 1, \dots n\} \subset D_2 = \{-1, 0, \dots\}.$$

Then $\mathrm{LEW}(0, D_2)$ is just the path $L_0 = 0, L_1 = -1$, while $\mathrm{LEW}(0, D_2)$ goes from 0 to n with probability $1/n$.

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Question. What is the maximum value

$$\frac{|\mathrm{LEW}(0, D_1^c)|}{|\mathrm{LEW}(0, D_2^c)|}$$

can take if $D_1 \subset D_2 \subset \mathbb{Z}^d$?

Point to ball and point to point

Recall $M_x = |\mathrm{LEW}(0, x)|$ and $M_n = |\mathrm{LEW}(0, B(0,n)^c)|$. Let $|x| = n + 1$.

1. The proof in MB, Masson 2010 (improved in MB, Croydon, Kumagai) actually gives that if $B(0,n) \subset D \subset \mathbb{Z}^2$ and $\widetilde{M}$ is the number of steps in $\mathrm{LEW}(0, D^c)$ between its first exit from $B(0, n/4)$ and its first exit from $B(0, 3n/4)$ then

$$\mathbb{P}(\widetilde{M} \le \lambda^{-1} n^{\kappa}) \le e^{-c\lambda^4}.$$

So proof of the upper bound on the tail works for both M_n and M_x.

2. The UST argument outlined in the talk gives

$$\mathbb{P}(M_x \le \lambda^{-1} n^{\kappa}) \ge e^{-c\lambda^4}.$$

Point to ball and point to point 2

3. To obtain a lower bound for $\mathbb{P}(M_n \leq \lambda^{-1} n^{\kappa})$ one has to work on a different graph: $\mathbb{Z}^2$ with all vertices outside $B(0,n)$ wired into a single point ∂_n.
Then one has

$$M_n =_{(d)} |\mathrm{LEW}(0, \partial_n)|,$$

and the same UST argument then gives the lower bound.