Mexican Hat coupling of quasi-cycle oscillators produces quasi-patterns

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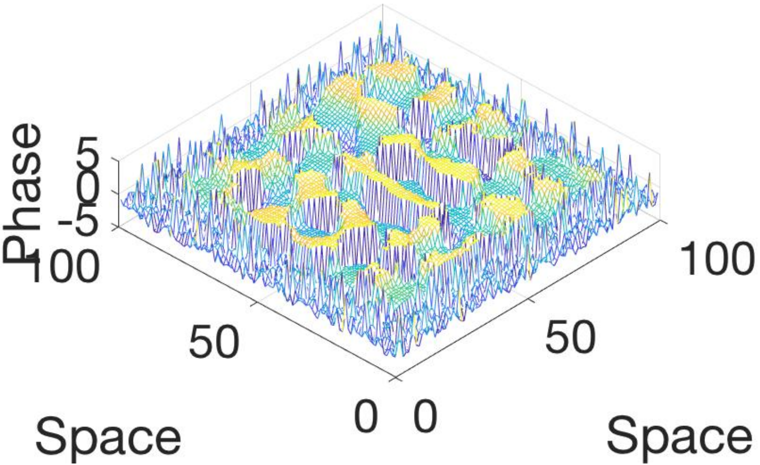
Stochastic Neural Field Equations



Stochastic Neural Field Equations

Stochastic reaction equation:

$$d_t \mathbb{Y}_j(t) = \mathbb{A}\mathbb{Y}_j(t)dt + d\mathbb{W}_j(t)$$

Eigenvalues of A:

$$-\lambda \pm i\omega$$

Stochastic neural field equation:

$$d_t \mathbb{Y}_j(t) = -\mathbb{Y}_j(t)dt + \mathbb{M}\mathbb{S}\mathbb{Y}_j(t)dt + \sigma d\mathbb{W}_j(t)$$

where

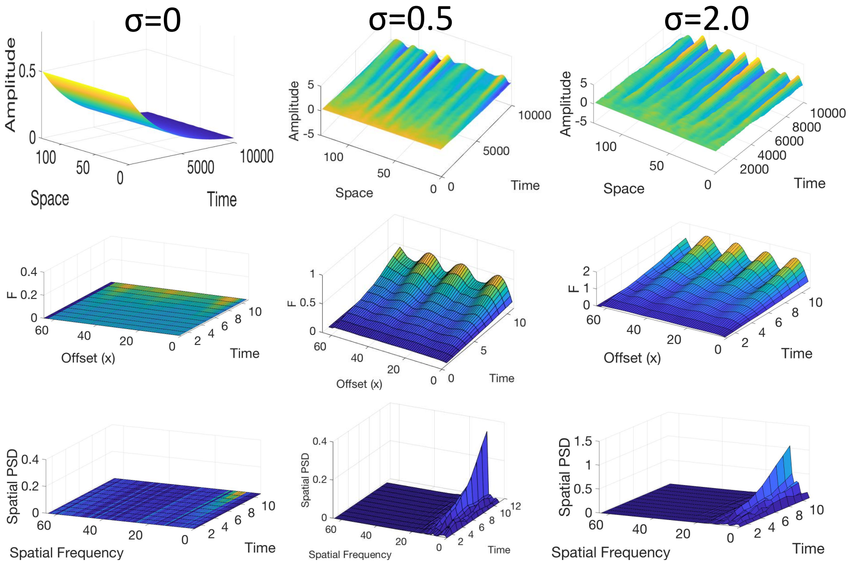
$$M\xi_j(t) = \sum cw(i-j)\xi_j(t),$$

$$w(x) = b_1 \exp\left[-\left(\frac{x}{d_1}\right)^2\right] - b_2 \exp\left[-\left(\frac{x}{d_2}\right)^2\right], \quad b_1 > b_2, d_2 > d_1,$$

and

$$S(Y) = \begin{cases} Y & \text{if } |Y| \leq Y_{max}, \\ 0.9Y & \text{if } |Y| > Y_{max}. \end{cases}$$

Stochastic Neural Field Equations



Stochastic Neural Field Equations

Stochastic reaction equation:

$$d_t \mathbb{Y}_j(t) = \mathbb{A}\mathbb{Y}_j(t)dt + d\mathbb{W}_j(t)$$

$$A = \begin{pmatrix} -\lambda & \omega \\ -\omega & -\lambda \end{pmatrix}$$

Stochastic neural field equation:

$$d_t \mathbb{Y}_j(t) = -\mathbb{Y}_j(t)dt + \mathbb{M}\mathbb{S}\mathbb{Y}_j(t)dt + d\mathbb{W}_j(t)$$

Stochastic reaction neural field equation:

$$d_t \mathbb{Y}_j(t) = \mathbb{A}\mathbb{Y}_j(t)dt + \mathbb{M}\mathbb{S}\mathbb{Y}_j(t)dt + d\mathbb{W}_j(t)$$

$$Y_j(t) = \begin{pmatrix} u_j(t) \\ v_j(t) \end{pmatrix}, \quad \theta_j = \arctan(v_j/u_j), \quad Z_j = (u_j^2 + v_j^2)^{1/2}$$

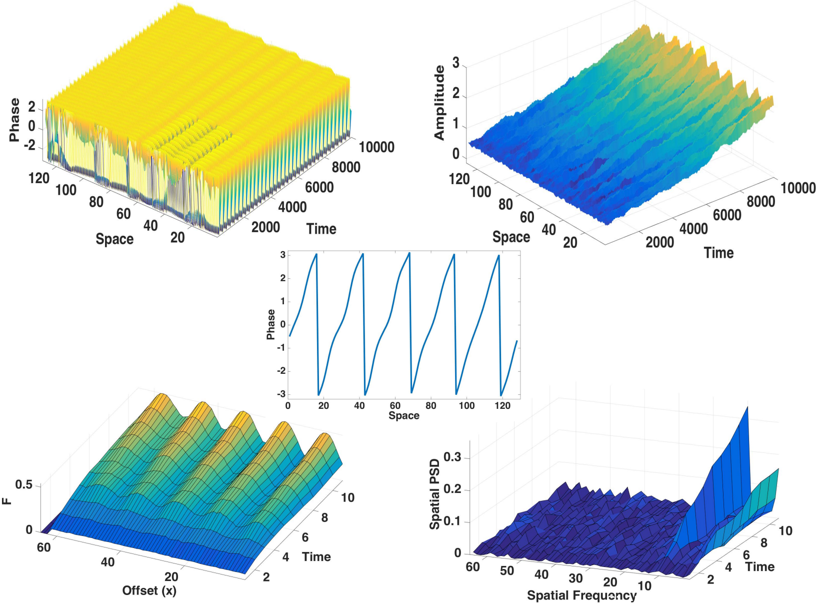
Stochastic Reaction-coupling Equations

Stochastic equations for phase and amplitude:

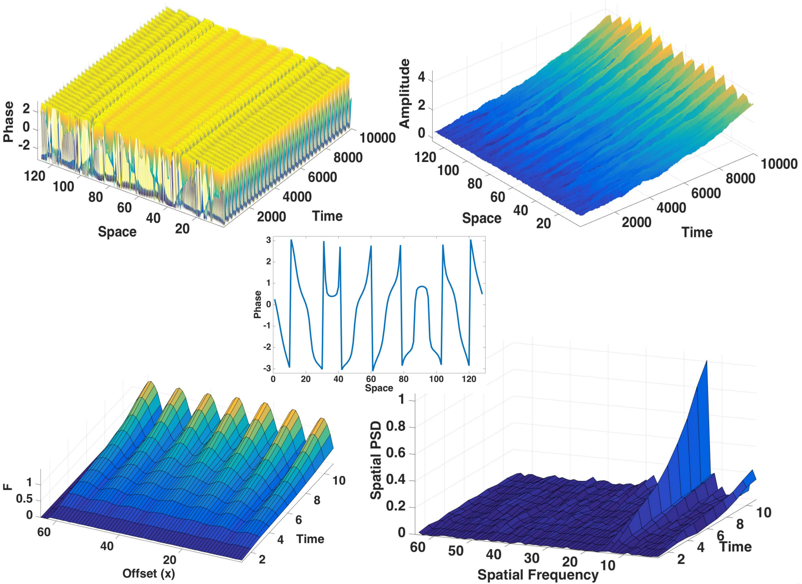
$$d\theta_j = \omega_j dt + \left[\sum_{l=1}^{N} \frac{Z_l(t)}{Z_j(t)} \mathbb{C}_{jl} \sin(\theta_j(t) - \theta_l(t)) \right] dt + \frac{db(t)}{Z_j(t)},$$

$$dZ_j = \left(\frac{1}{2Z_j(t)} - \lambda Z_j(t) \right) dt + \left[\sum_{l=1}^{N} \mathbb{C}_{jl} Z_j cos(\theta_j(t) - \theta_l(t)) \right] dt + dW_j(t),$$

Stochastic Reaction-coupling Equationss



Stochastic Reaction-coupling Equations



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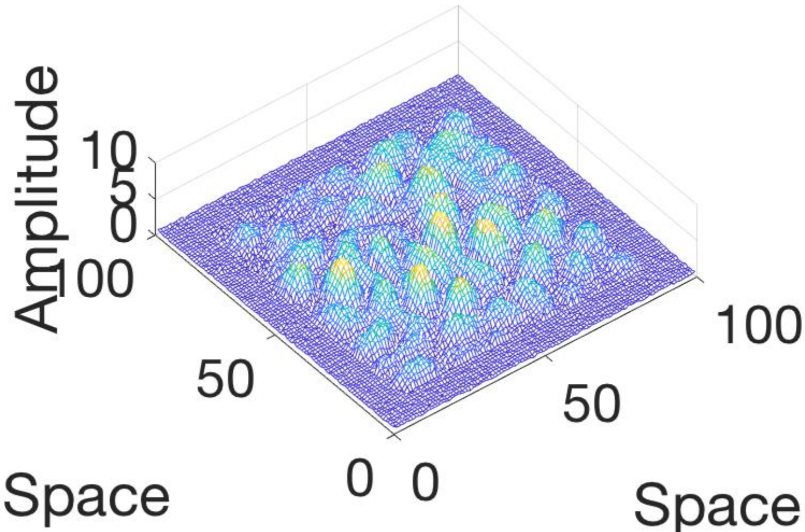

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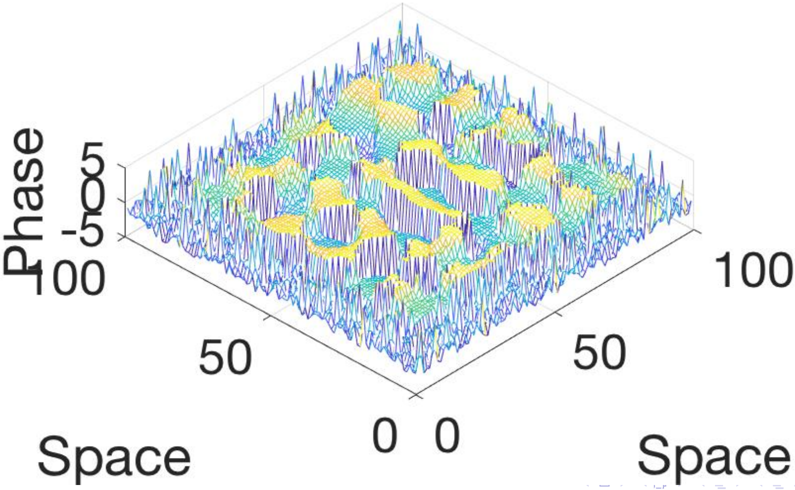
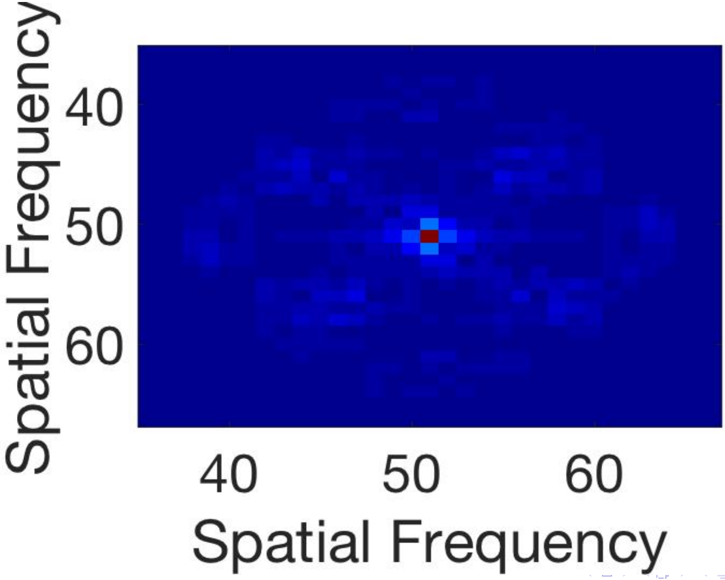


Stochastic Reaction-coupling Equations



Approximate Factorization of Stochastic Reaction-coupling Process

$$d_t\mathbb{Y}(t) = \mathbb{A}\mathbb{Y}(t)dt + d\mathbb{W}(t)$$

Theorem of Baxendale and Greenwood (2011): if λ/ω is small

$$\mathbb{Y}(t) \approx \tilde{\mathbb{Y}}(t) := \frac{\sigma}{\sqrt{\lambda}}\mathbb{R}_{-\omega t}\mathbb{U}(\lambda t),$$

where

$$\mathbb{R}_s = \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix}$$

$\mathbb{U}(t)$ is a pair of independent Ornstein-Uhlenbeck processes

$$d\mathbb{U}(t) = -\mathbb{U}(t)dt + dW(t)$$

Approximate Factorization of Stochastic Reaction-coupling Process

The phase and amplitude processes of

$$\tilde{\mathbb{Y}}(t) = \frac{\sigma}{\sqrt{\lambda}} \mathbb{R}_{-\omega t} \mathbb{U}(\lambda t)$$

are

$$d\theta = -\omega dt + d\phi(\lambda t)$$

where ϕ satisfies

$$d\phi(t) = \frac{db(t)}{Z(t)},$$

$$Z(t) = \frac{\sigma}{\sqrt{\lambda}} \bar{Z}(\lambda t)$$

where $\bar{Z}$ satisfies

$$d\bar{Z}(t) = \left(\frac{1}{2\bar{Z}(t)} - \bar{Z}(t) \right) dt + dW(t).$$

Approximate Factorization of Stochastic Reaction-coupling Process

Stochastic reaction neural field with balanced terms:

$$d_t \mathbb{Y}_j(t) = \mathbb{A}\mathbb{Y}_j(t)dt + \frac{\lambda}{\omega}\mathbb{M}\mathbb{S}\mathbb{Y}_j(t)dt + d\mathbb{W}_j(t).$$

New approximate factorization:

$$\mathbb{Y}_j(t) \approx \tilde{\mathbb{Y}}_j(t) = \frac{\sigma}{\sqrt{\lambda}}\mathbb{R}_{-\omega t}\mathbb{U}_j^*(\lambda t),$$

where $\mathbb{U}_j^*$ satisfies

$$d_t \mathbb{U}_j^*(t) = -\mathbb{U}_j^*(t)dt + \mathbb{M}\mathbb{U}_j^*(t)dt + d\mathbb{W}_j(t),$$

i.e., $\mathbb{U}_j^*$ is a stochastic neural field.

Approximate Factorization of Stochastic Reaction-coupling Process

The phase and amplitude processes of

$$\tilde{\mathbb{Y}}_j(t) = \frac{\sigma}{\sqrt{\lambda}} \mathbb{R}_{-\omega t} \mathbb{U}_j^*(\lambda t),$$

are

$$\theta_j(t) = -\omega t + \phi(\lambda t),$$

where $\phi(t)$ satisfies

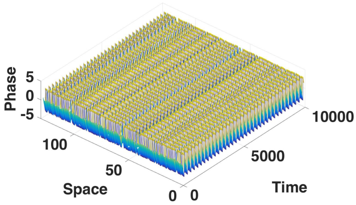
$$d\phi_j(t) = \left[\sum_{l=1}^{N} \frac{Z_l(t)}{Z_j(t)} \mathbb{C}_{jl} \sin(\phi_j(t) - \phi_l(t)) \right] dt + \frac{db(t)}{Z_j(t)},$$

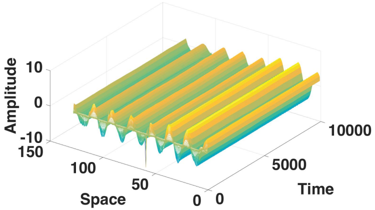
$$Z_j(t) = \frac{\sigma}{\sqrt{\lambda}} \bar{Z}(\lambda t)$$

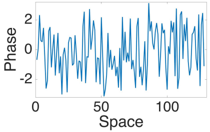
where $\bar{Z}(t)$ satisfies

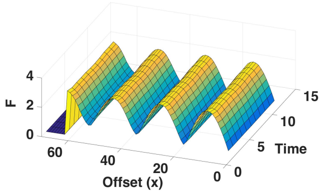
$$d\bar{Z}_j = \left(\frac{1}{2Z_j(t)} - Z_j(t) \right) dt + \left[\sum_{l=1}^{N} \mathbb{C}_{jl} Z_j cos(\phi_j(t) - \phi_l(t)) \right] dt + dW_j(t).$$

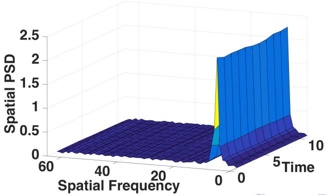
Approximate Factorization of Stochastic Reaction-coupling Process











Thank you!