Question 5 solutions (2013 Q6)

(a) Using de Moivre's Theorem, and collecting real and imaginary terms:

$$\cos 5\theta + \mathrm{i}\sin 5\theta = \mathrm{cis}\, 5\theta = \mathrm{cis}^5\theta$$
$$= (\cos\theta + \mathrm{i}\sin\theta)^5$$
$$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\mathrm{i}\sin 5\theta = 5\mathrm{i}\cos^4\theta\sin\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + \mathrm{i}\sin^5\theta$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

Alternatively, with rather more work, the sum of angle formulas can get the same results. Differentiating either identity gives the other.

6(a) Use de Moivre's Theorem [1st mark]
expand (binomial theorem or otherwise [2nd mark]
simplify, separate real / imaginary parts [3rd mark]
well reasoned proof [4th mark].

(b)

EITHER

Using $x_{n+1} = x_n, y_{n+1} = y_n$ and $z_{n+1} = z_n$, we get the equations

$$x_n = 0.8x_n + 0.7y_n + 0.6z_n$$
$$y_n = 0.1x_n + 0.2y_n + 0.4z_n$$
$$z_n = 0.1x_n + 0.1y_n$$

yielding:

$$-0.2x_n + 0.7y_n + 0.6z_n = 0$$
$$0.1x_n - 0.8y_n + 0.4z_n = 0$$
$$0.1x_n + 0.1y_n - z_n = 0$$

We also need the additional information that $x_n + y_n + z_n = 99$, as the first three are insufficient.

Using any of several ways to solve this system of four equations, we find that

$x_n = 76$
$y_n = 14$
$z_n = 9$

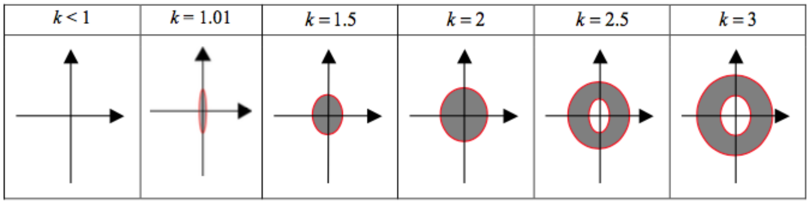
6(a) *linear systems*
Set $x_{n+1} = x_n$ etc [1st mark] collect terms [2nd mark]
introduce extra equation $x + y + z = 99$ [3rd mark]
solution $(x, y, z) = (76, 14, 9)$ [4th mark]

OR

Consider the equation $|z+\mathrm{i}| + |z-\mathrm{i}| = A$. This is an ellipse, with foci at i and $-\mathrm{i}$, so long as $A > 2$. When $A = 2$ it is the set of points on the imaginary axis between i and $-\mathrm{i}$.

If $k < 1$ then there are no such points.
At $k = 1$ the points are on the imaginary axis between i and $-\mathrm{i}$.
If $1 < k \le 2$ the points form a solid ellipse, with foci at i and $-\mathrm{i}$.
If $k > 2$ the points lie between two ellipses with foci at i and $-\mathrm{i}$; the outer ellipse has an inner ellipse of points which do not satisfy the inequality.
(For large k, both ellipses are approximately circles, with the inner circle half the radius of the outer.)

$k < 1$	$k = 1.01$	$k = 1.5$	$k = 2$	$k = 2.5$	$k = 3$



6(a) *conic sections*
Form of ellipse with foci correct [1st mark]
ring shape [2nd mark], solid when $1 \le k \le 2$ [3rd mark],
no solutions when $0 < k < 1$ [4th mark].