Some Identities Concerning Excursion Below Zero with Fixed Duration of Spectrally Negative Lévy Process

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Problem Motivation: Cramér-Lundberg Model

Consider the classical **Cramér-Lundberg model** of surplus process

$$X_t = x + ct - \sum_{i=1}^{N_t^\lambda} \xi_i, \quad \text{with} \quad \xi_i \sim F. \tag{1}$$

The Laplace exponent ψ of the Cramér-Lundberg process (1) is

$$\psi(\theta) = c\theta - \lambda \int_{(0,\infty)} \big(1 - e^{-\theta x}\big) F(dx). \tag{2}$$

In order to avoid ruin occurring with probability one, we assume that $\psi'(0+) > 0$.

Default/ruin of the insurance company is announced at the stopping time

$$\tau_0^- = \{t > 0 : X_t < 0\}. \tag{3}$$

The most basic question one may want to find is the probability of ruin

$$\mathbb{P}_x\{\tau_0^- < \infty\}. \tag{4}$$

Pollaczek-Khintchine formula, Emery Identity and $q-$resolvent density

Theorem 1. [Pollaczek-Khintchine] *Suppose that $\psi'(0+) > 0$. For all $x \geq 0$,*

$$\mathbb{P}_x\{\tau_0^- < \infty\} = 1 - (1-\rho)\sum_{k=0}^{\infty} \rho^k \eta^{*k}(x), \tag{5}$$

where

$$\rho = \frac{\lambda}{c}\int_0^\infty xF(dx) \quad \text{and} \quad \eta(x) = \frac{\int_0^x [1-F(y)]dy}{\int_0^\infty yF(dy)},$$

where $\eta^{\star^k}$ is the $k-$fold convolution of η with $\eta^{\star^0}(dx) := \delta_0(dx)$.

Theorem 2. [Emery fluctuation identity] *For any $q, \theta \geq 0$, then for all $x \geq 0$,*

$$\begin{aligned}\mathbb{E}_x\left\{e^{-q\tau_0^- + \theta X_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \infty\}}\right\} =& e^{\theta x} - \frac{(q-\psi(\theta))}{(\Phi(q)-\theta)} W^{(q)}(x) \\ &+ (q-\psi(\theta))e^{\theta x}\int_0^x e^{-\theta y} W^{(q)}(y)dy,\end{aligned} \tag{6}$$

where $W^{(q)}(x)$ is defined by $\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta)-q}$, for $q \geq 0, \theta > \Phi(q)$.

Pollaczek-Khintchine formula, Emery Identity and $q-$resolvent density

Remarks 1. *By setting $q = 0 = \theta$ in the Emery identity, we arrive at the Pollaczek-Khintchine formula (5) taking account of the fact that*

$$\psi'(0+) = c - \lambda \int_0^\infty yF(dy),$$

and the scale function corresponding to the Cramér-Lundberg process (1) is

$$W(x) = \frac{1}{c} \sum_{k=0}^{\infty} \rho^k \eta^{*k}(x).$$

Lemma 1. [$q-$resolvent density, Bertoin [Be97]] *Denote by $\mathbf{e}_q$ an independent exponential random variable with mean q^{-1}. We have for every $x \geq y$ and $q > 0$ that*

$$q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \tau_0^-\} = \big(e^{-\Phi(q)y}W^{(q)}(x) - W^{(q)}(x-y)\big)dy.$$

Pollaczek-Khintchine formula, Emery Identity and $q-$resolvent density

Emery's identity and the $q-$resolvent measure have appeared in some places, e.g.,

- *scattering theory* (Fourati (2010)),
- *credit risk management* (Egami & Yamazaki (2013), Egami and Oryu (2015)),
- *ruin/dividend problems* (Chiu & Yin (2005), Jacobsen (2005), Zhou (2004), Bratiichuk (2012), Frostig (2015), Landriault et al (2011)),
- *capital structure* (Kyprianou & **Surya** (2008), **Surya** and Yamazaki (2014)),
- *queueing problem* (Dube et al (2004), Zwart (2015), Bekker et al (2008), Bekker (2009)),
- *boundary problem* (Kadankova & Veraverbeke (2007)), etc
- *branching problem* (Lambert & Trapman (2013), Konstantapolous et al (2011)),
- *contraction options pricing* Yamazaki (2015)
- *epidemics problem* Lambert et al. (2014)
- *fragmentation problem* Krell (2008), Krell and Rouault (2011)

Excursion Below Zero of SNLP with Fixed Duration

The main object of study is the quantity τ_r with $r > 0$ called the $Parisian\ ruin\ time$:

$$\tau_r = \inf\{t > 0 : \mathbf{1}_{\{X_t<0\}}(t - g_t) \geq r\}, \tag{7}$$

with

$$g_t := \sup\{s < t : X_s \geq 0\}.$$

representing the first time that the Lévy process X has spent $r > 0$ units of time consecutively below zero before getting back up to zero again.

Remarks 2. *When $r = 0$, ruin leads to an immediate liquidation which takes place the first time the surplus process X goes below zero, i.e., at the stopping time τ_0^-.*

The stopping time τ_r (7) was first introduced by Chesney et al. [Ch97] in the context of pricing barrier options in mathematical finance. It was later introduced in actuarial risk theory by Dassios and Wu [Da09a] under the classical Cramer-Lundberg surplus process (1) and provided an expression for the $Parisian\ ruin\ probability$

$$\mathbb{P}_x\{\tau_r < \infty\}. \tag{8}$$

The Parisian Ruin Probability

Czarna and Palmowski [Cz11] and Loeffen et al. [Lo13] extended the result to the class of spectrally negative Lévy processes in which the CL surplus model (1) is included.

The Parisian ruin probability derived in [Da09a], [Cz11], [Lo13] can be summarized as

$$\mathbb{P}_x\{\tau_r < \infty\} = 1 - \mathbb{E}\{X_1\}\frac{\int_0^\infty W(x+z)z\mathbb{P}\{X_r \in dz\}}{\int_0^\infty z\mathbb{P}\{X_r \in dz\}},$$

where $W(x)$ is the scale function explained before:

$$\int_0^\infty e^{-\theta x} W(x) dx = \frac{1}{\psi(\theta)}, \quad \theta > 0.$$

with $W(x) = 0$ for $x < 0$.

Problems and The Main Results

Our main object of interest

We consider the joint Laplace transform of τ_r and X_{τ_r} and the resolvent measure

$$\mathbb{E}_x\Big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\infty\}}\Big\} \quad \text{and} \quad q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q}\in dy, \mathbf{e}_q<\tau_r\}, \tag{9}$$

for $q,\theta\geq 0$ and all $x,y\in\mathbb{R}$. We recently attempted to give a semi explicit form

$$\mathbb{E}_x\Big\{\int_0^{\tau_r} e^{-qt}f(X_t)dt + e^{-q\tau_r}g(X_{\tau_r})\Big\}, \tag{10}$$

where f and g are non-negative Borel measurable payoff functions. The result gives a generalization of Baurdoux et al. [Ba14] under randomized excursion length.

The results obtained are available in semi-explicit forms in terms of the $q-$scale function $W^{(q)}(x)$ and the law $\mathbb{P}\{X_t\in dx\}$ of the Lévy process.

- We show in the limit as the excursion duration r goes to zero that the joint Laplace transform leads to Emery's fluctuation identity for first exit below zero, whereas the $q-$resolvent kernel leads to a resolvent measure given by Bertoin (1997).

Spectrally Negative Lévy Process

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$. That is to say that

- X is a stochastic process starting from zero, having stationary and independent increments with cádlág (RCLL) sample paths with no positive jumps.

As a strong Markov process, we shall endow X with probabilities $\{\mathbb{P}_x, x \in \mathbb{R}\}$, such that $\mathbb{P}_x\{X_0 = x\} = 1$. Further, we denote by $\mathbb{E}_x$ expectation with respect to $\mathbb{P}_x$. Recall that $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$. Due to the absence of positive jumps, we have

$$\psi(\lambda) = \frac{1}{t}\log \mathbb{E}\{e^{\lambda X_t}\} = \mu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \left(e^{\lambda y} - 1 - \lambda y \mathbf{1}_{\{y>-1\}}\right)\Pi(dy).$$

which is analytic on $(\mathfrak{Im}(\lambda) \leq 0)$, with $\mu \in \mathbb{R}$ and $\sigma \geq 0$. It is easily shown that

- ψ is zero at the origin, tends to infinity at infinity and is strictly convex.

Spectrally Negative Lévy Process: Continued

The class of spectrally negative Lévy processes is very rich. Among other things it allows for the processes to have paths of **unbounded variation** if and only if

$$\sigma > 0 \quad \text{or} \quad \int_{(-1,0)} |x|\Pi(dx) = \infty.$$

Otherwise, it has **bounded variation** paths if and only if

$$\sigma = 0 \quad \text{and} \quad \int_{(-\infty,0)} |x|\Pi(dx) < \infty,$$

which in that case one may rearrange the Laplace exponent ψ into the form

$$\psi(\theta) = \mathrm{d}\theta - \int_{(-\infty,0)} \left(1 - e^{\theta x}\right)\Pi(dx),$$

where necessarily $\mathrm{d} > 0$. This reflects the fact that a spectrally negative of bounded variation constitutes the difference of a linear drift and a pure jump subordinator.

For background and further details on spectrally negative Lévy process, we refer to *Chapter VI of Bertoin (1996)* and *Chapter 8 of Kyprianou (2006)*.

Spectrally Negative Lévy Process: Continued

We denote by $\Phi : [0,\infty) \to [0,\infty)$ the right continuous inverse of ψ so that

$$\Phi(\theta) = \sup\{p > 0 : \psi(p) = \theta\} \quad \text{and} \quad \psi(\Phi(\lambda)) = \lambda \quad \text{for all} \quad \lambda \geq 0.$$

- By convexity of ψ: $\exists$ two roots for a given θ and precisely one root when $\theta > 0$.
- The asymptotic behavior of X can de determined from the sign of $\psi'(0+)$,
  - X drifts to $-\infty$, oscillates or drifts to $+\infty$, i.e., $\mathbb{P}\{\lim_{t\to\infty} X_t = \infty\} = 1$ according to whether $\psi'(0+)$ is < 0, 0, or > 0.

It is worth mentioning that under the **Esscher transform of measure** $\mathbb{P}^\nu$ defined by

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = e^{\nu X_t - \psi(\nu)t} \quad \text{for all } \nu \geq 0, \tag{11}$$

the Lévy process $(X, \mathbb{P}^\nu)$ is still a spectrally negative Lévy process.

Spectrally Negative Lévy Process: Continued

The Laplace exponent of X under the new measure $\mathbb{P}^\nu$ has changed to

$$\begin{aligned}\psi_\nu(\lambda) &= \psi(\lambda+\nu)-\psi(\nu)\\ &= \Big(\mu+\sigma^2\nu+\int_{(-\infty,0)} y\big(e^{\nu y}-1\big)\mathbf{1}_{\{y>-1\}}\Pi(dy)\Big)\lambda \qquad (12)\\ &+\frac{1}{2}\sigma^2\lambda^2+\int_{(-\infty,0)}\big(e^{\lambda y}-1-\lambda y\mathbf{1}_{\{y>-1\}}\big)e^{\nu y}\Pi(dy), \quad \text{for } \lambda\geq-\nu\end{aligned}$$

To each $\nu \geq 0$, we will denote by $\mathbb{P}^\nu_x$ the translation of $\mathbb{P}^\nu$ under which $X_0 = x$.

Subsequently, we define by $\Phi_\nu(\theta)$ the largest root of $\psi_\nu(\lambda) = \theta$ satisfying

$$\Phi_\nu(\theta) = \Phi(\theta+\psi(\nu)) - \nu.$$

- Note under the new measure $\mathbb{P}^\nu$, we have that $\psi_\nu(0+) > 0$.

Spectrally Negative Lévy Process: Continued

Associated with the Laplace exponent ψ is an increasing $q-$**scale function**

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \quad \text{for } q \geq 0, \theta > \Phi(q), \tag{13}$$

with $W^{(q)}(x) = 0$ for $x < 0$. We shall write for short $W^{(0)} = W$ and refer to $W_\nu^{(q)}$ the scale function under $\mathbb{P}^\nu$. Following (13), it is straightforward to check that

$$W_\nu^{(q)}(x) = e^{-\nu x} W^{(q+\psi(\nu))}(x) \quad \text{for all } \nu \geq 0 \text{ and } q \geq -\psi(\nu).$$

In the sequel below, we will use the notation $\overline{W}_\nu^{(q)}(x) = \int_0^x W_\nu^{(q)}(y)dy$.

- Some examples of Lévy processes for which the scale function is available in explicit form can be found in Kuznetzov et al. [Ku13a].
- In any case, $W^{(q)}(x)$ can be computed by numerically by inverting the Laplace transform (13) under new change of measure $\mathbb{P}^{\Phi(q)}$, see **Surya** [Su14].

The Main Results

Theorem 3. *Assume that $\psi'(0+) > 0$. Then for any $q \geq 0$,*

$$\begin{aligned}\mathbb{E}_x\left\{e^{-q\tau_r}\mathbf{1}_{\{\tau_r<\infty\}}\right\} =& e^{-qr}\Big\{1 + q\int_0^x W^{(q)}(z)dz + \frac{q}{r}\int_0^r A^{(q)}(x,u)du \\ & - \frac{\frac{q}{\Phi(q)} + \frac{q}{r}\int_0^r\int_0^\infty e^{\Phi(q)z} z\mathbb{P}\{X_u \in dz\}du}{\int_0^\infty e^{\Phi(q)z} z\mathbb{P}\{X_r \in dz\}} A^{(q)}(x,r)\Big\},\end{aligned} \tag{14}$$

for all $x \in \mathbb{R}$, where the function $A^{(q)}(x,r)$ is defined by

$$A^{(q)}(x,r) := \int_0^\infty W^{(q)}(x+z) z\mathbb{P}\{X_r \in dz\}.$$

Remarks 3. *Setting $q = 0$ in (14), we obtain the Parisian ruin probability*

$$1 - \mathbb{P}_x\{\tau_r < \infty\} = \frac{\psi'(0+)\int_0^\infty W(x+z) z\mathbb{P}\{X_r \in dz\}}{\int_0^\infty z\mathbb{P}\{X_r \in dz\}}. \tag{15}$$

The Main Result

Under Esscher transform of measure $\mathbb{P}^\theta$, we derive the joint Laplace transform:

Corollary 1. *Let $p = q - \psi(\theta)$. Then, for any $q, \theta \geq 0$ with $q > \psi(\theta)$,*

$$\mathbb{E}_x\Big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\infty\}}\Big\} = e^{-pr+\theta x}\Big\{1 + p\overline{W}_\theta^{(p)}(x) + \frac{p}{r}\int_0^r A_\theta^{(p)}(x,u)du - \frac{\left(\frac{p}{\Phi_\theta(p)} + \frac{p}{r}\int_0^r\int_0^\infty e^{\Phi_\theta(p)z} z\mathbb{P}^\theta\{X_u \in dz\}du\right)}{\int_0^\infty e^{\Phi_\theta(p)z} z\mathbb{P}^\theta\{X_r \in dz\}} A_\theta^{(p)}(x,r)\Big\},$$

for all $x \in \mathbb{R}$, where the function $A_\theta^{(p)}(x,r)$ is defined by

$$A_\theta^{(p)}(x,r) := \int_0^\infty W_\theta^{(p)}(x+z) z\mathbb{P}^\theta\{X_r \in dz\}.$$

Theorem 4. *For any $q, y \geq 0$, we have for all $x \in \mathbb{R}$ that*

$$q^{-1}\mathbb{P}_x\{X_{\mathbf{e}_q} \in dy, \mathbf{e}_q < \tau_r\} = \Big(\frac{e^{\Phi(q)(x-y)} A_{\Phi(q)}^{(0)}(x,r)}{\int_0^\infty z\mathbb{P}^{\Phi(q)}\{X_r \in dz\}} - W^{(q)}(x-y)\Big)dy.$$

The Main Result

By sending the length of excursion r goes to zero, we obtain that

$$\lim_{r\downarrow 0}\mathbb{E}_x\Big\{e^{-q\tau_r+\theta X_{\tau_r}}\mathbf{1}_{\{\tau_r<\infty\}}\Big\} = e^{\theta x}-\frac{(q-\psi(\theta))}{(\Phi(q)-\theta)}W^{(q)}(x) + (q-\psi(\theta))e^{\theta x}\int_0^x e^{-\theta y}W^{(q)}(y)dy,$$

which coincides with Emery's fluctuation identity (6). Whilst,

$$\lim_{r\downarrow 0} q^{-1}\mathbb{P}_x\big\{X_{\mathbf{e}_q}\in dy, \mathbf{e}_q<\tau_r\big\} = \Big(e^{-\Phi(q)y}W^{(q)}(x)-W^{(q)}(x-y)\Big)dy.$$

which is exactly the $q-$resolvent density of X found in Bertoin [Be97]:

Preliminaries

We frequently use the **Kendall's identity** (*Corollary VII.3* in [Be96])

$$t\mathbb{P}\{\tau_x^+ \in dt\}dx = x\mathbb{P}\{X_t \in dx\}dt. \tag{16}$$

Lemma 2. *For any $\theta, q \geq 0$ with $\theta > q$, we have for all $x \geq 0$,*

$$\begin{aligned}\mathbb{E}_x\Big\{e^{-q\tau_0^- + \Phi(\theta)X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^- < \infty\}}\Big\} =& \frac{(\theta - q)}{\Phi(\theta)}\int_0^\infty e^{-\Phi(\theta)y} W^{(q)\prime}(x+y)dy \\ &+ \Big(\frac{\theta - q}{\Phi(\theta)} - \frac{\theta - q}{\big(\Phi(\theta) - \Phi(q)\big)}\Big)W^{(q)}(x).\end{aligned} \tag{17}$$

Lemma 3. *For $y > 0$ and $\theta > 0$ such that $0 \leq \alpha < \Phi(\theta)$,*

$$\int_0^\infty e^{-\theta r} e^{-\alpha y}\int_y^\infty e^{\alpha z}\frac{z}{r}\mathbb{P}\{X_r \in dz\}dr = \frac{e^{-\Phi(\theta)y}}{\big(\Phi(\theta) - \alpha\big)}, \tag{18}$$

$$\int_0^\infty e^{-\theta r} e^{-\alpha y}\int_y^\infty e^{\alpha z}\mathbb{P}\{\tau_z^+ \leq r\}dzdr = \frac{e^{-\Phi(\theta)y}}{\theta\big(\Phi(\theta) - \alpha\big)}. \tag{19}$$

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Thank you !