Bootstrap random walks

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- We study the (time-inhomogeneous) Markov process $W_n = (X_n, Y_n)$.

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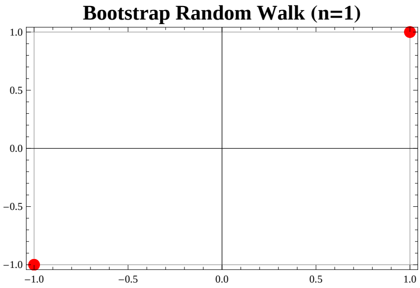

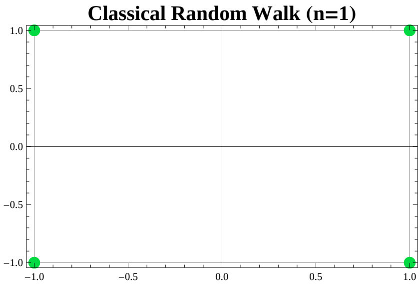

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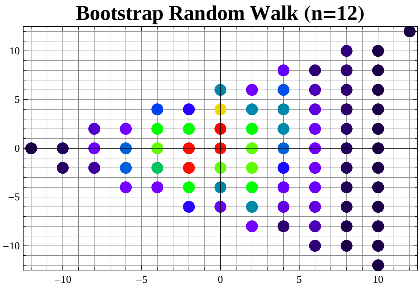

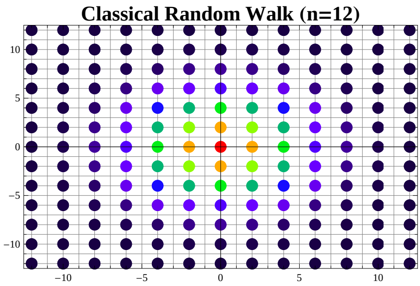

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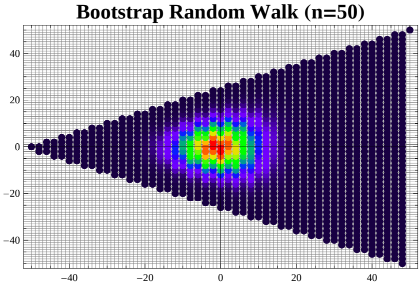
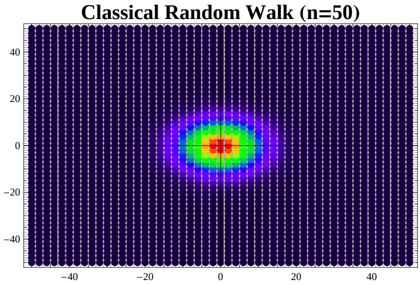
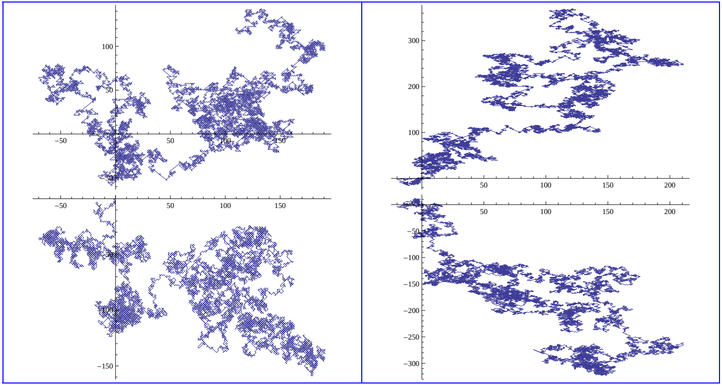




Long-term behaviour

- Long-term, $(W_n)_n$ behaves like a two dimensional random walk.



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Theorem

$W_n = (X_n, Y_n)$ is recurrent.

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- We consider the three-dimensional random walk (X_n, Y_n, Z_n), also denoted W_n, where $Z_n = \sum_{k=1}^{n} \zeta_k$, $n \geq 1$, and $Z_0 = 0$.

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Proposition

1. For any $n \geq 2$,

$$\mathbb{P}(W_{4n}=0) = 2^{-4n}\sum_{k=0}^{n-2}\binom{n-1}{k}\binom{n}{k+1}\binom{n}{k+1}\binom{n-1}{k+1}$$

and

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2. $\mathbb{P}(W_{2n}=0)=O(n^{\alpha-2})$, for any $\alpha\in(1/2,1)$.
3. $(W_n)_n$ is transient; it will visit the origin finitely often.

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Theorem

$\mathfrak{W}_n(t) = (\mathfrak{X}_n(t), \mathfrak{Y}_n(t), \mathfrak{Z}_n(t))$ converges weakly to a three-dimensional Brownian motion.

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- $\eta_{0,n} = \xi_n$, $\eta_{1,n} = \eta_n = \prod_{\ell=1}^{n} \xi_\ell$ and $\eta_{\kappa+1,n} = \prod_{\ell=1}^{n} \eta_{\kappa,\ell}$.

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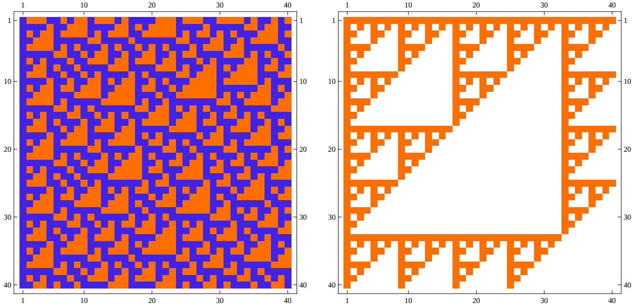

Figure : A graphical representation of $\eta_{\kappa,n}$ (left) and $\nu_{\kappa,n}$ (right)

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- $\mathfrak{Y}_{k,n}(t) = \dfrac{1}{\sqrt{n}} Y_{k,\lfloor nt \rfloor}$ converge weakly to a Brownian motion $(t \in [0,1])$.
- We look at the limiting behaviour of $\mathfrak{W}_n(t) = (\mathfrak{Y}_{0,n}(t), \dots, \mathfrak{Y}_{\kappa,n}(t))$.
- Must know which ξ_j's in $\eta_{k,n} = \displaystyle\prod_{\ell=1}^{n} \xi_{n-\ell+1}^{\nu_{k,\ell}}$ are "switched on".

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$$\binom{93}{77} = 380813488615359423 \text{ is odd:}$$

93	1011101_2
77	1001101_2

.

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$$\binom{73}{25} = 23214764053299962052 \text{ is even:}$$

73	$10\textcolor{red}{0}1001_2$
25	$00\textcolor{red}{1}1001_2$

.

Three-value model

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- This can be repeated and generalised.

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- $(\eta_{\kappa,n})_n$ has the same distribution as $(\xi_n)_n$.
- Let $Y_{\kappa,n} = \sum_{\ell=1}^{n} \eta_{\kappa,\ell}$, with $Y_{\kappa,0} = 0$.

The central limit theorem

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The vector $(\eta_{0,n+\kappa}, \ldots, \eta_{\kappa,n+\kappa})$ is uniform over $\mathcal{U}^{\kappa+1}$ and is independent of $\mathcal{F}_{n-1}$. In particular, $\eta_{0,n+\kappa}, \ldots, \eta_{\kappa,n+\kappa}$ are independent.

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Let $\sigma^2 = \mathbb{E}[\xi_n^2] = \dfrac{1}{p}(u_0^2+\cdots+u_{p-1}^2)$ and $\mathfrak{Y}_{\kappa,n}(t) = \dfrac{1}{\sigma\sqrt{n}} Y_{\kappa,\lfloor nt\rfloor}$.

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Theorem

$\mathfrak{W}_{\kappa,n}(t) = (\mathfrak{X}_n(t), \mathfrak{Y}_{1,n}(t)), \ldots, \mathfrak{Y}_{\kappa,n}(t))$ converges weakly to a $(\kappa+1)$-dimensional Brownian motion.

- Let $\eta_n = \phi_n(\xi_1, \ldots, \xi_n)$. Look for $(\phi_n)_n$ s.t. $(\eta_n)_n \stackrel{d}{=} (\xi_n)_n$.

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$(\eta_n)_n \overset{d}{=} (\xi_n)_n$ if and only if $\phi_n(x_1, \ldots, x_n)$ is of the following form:

$$\phi_1(x_1) = \pm x_1, \text{ for } n \geq 2, \ \phi_n(x_1, \ldots, x_n) = \psi_n(x_1, \ldots, x_{n-1}) x_n,$$

where ψ_n is any function from $\{-1, +1\}^{n-1}$ to $\{-1, +1\}$.

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where ψ_n is any function from $\{-1, +1\}^{n-1}$ to $\{-1, +1\}$.

- Let $\mathbb{K}(n)$ be the set of all non-empty subsets of $\{1, \dots, n\}$.

$$\psi_n(x_1, \dots, x_{n-1}) = (-1)^{\alpha_n} \left(\prod_{K \in \mathbb{K}(n-1)} x_{[K]}^{\beta_{n,K}} \right),$$

where $x_{[K]} = \max_{k \in K} x_k$ and $\alpha_n, \beta_{n,K} \in \{0, 1\}$.

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Theorem

Suppose that there exists $m \geq 1$ such that $(\xi_{n+m+1}, \eta_{n+m+1})$ is independent of $(\xi_1, \dots, \xi_n)$. Let $\rho = \mathbb{E}[\xi_n \eta_n]$. Then $(\mathfrak{X}_n(t), \mathfrak{Y}_n(t))$ converges weakly to a two-dimensional Brownian motion (possibly degenerate) with correlation ρ.

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- Sufficient conditions for the independence of $(\xi_{n+m+1}, \eta_{n+m+1})$ and $(\xi_1, \ldots, \xi_n)$ can be obtained.

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- Sufficient conditions for the independence of $(\xi_{n+m+1}, \eta_{n+m+1})$ and $(\xi_1, \ldots, \xi_n)$ can be obtained.
- $\beta_{n+m+1,K} = 0$ whenever $K \in \mathbb{K}(n)$ is sufficient. Eg

$$\eta_{n+2} = \max(\xi_n, \xi_{n+1})\xi_{n+2}.$$

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- Let $\Gamma_m(p) = \left\{\gamma \in \{0,1\}^{\mathbb{K}(m)}; \mathbb{P}\left(\prod_{K \in \mathbb{K}(m)} \xi_{[K]}^{\gamma(K)} = -1\right) = p\right\}$.

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- Any $\gamma \in \Gamma_{m+1}(p)$ can be uniquely constructed from a $\gamma_1 \in \Gamma_m(p_1)$ and $\gamma_2 \in \Gamma_m(p_2)$, where $(p_1 + p_2)/2 = p$.

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- When $m = 1$, independence ocurs if and only if either $\eta_{n+2} = (-1)^{\alpha_{n+2}}\xi_{n+2}$ (degenerate case) or

$$\eta_{n+2} = (-1)^{\alpha_{n+2}}\left(\prod_{K\in\mathbb{K}(n)} \xi_{[K]}^{\beta_{n+2,K}}\right)\xi_{n+1}\xi_{n+2}.$$

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- Let $\gamma(K) = \beta_{n+m+1,K+n}$. If $\gamma \in \Gamma_m(1/2)$, then independence holds and, ξ_{n+m+1} and η_{n+m+1} are independent and uniform.

A related question – percolation (work in progress)

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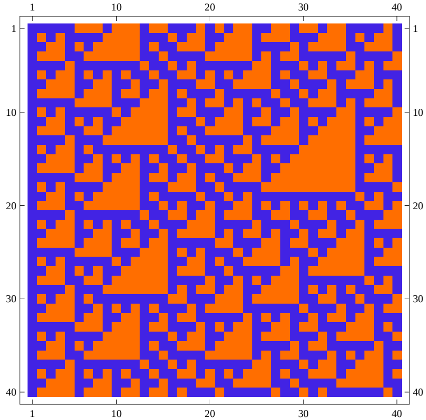

Figure : $p = 0.5$

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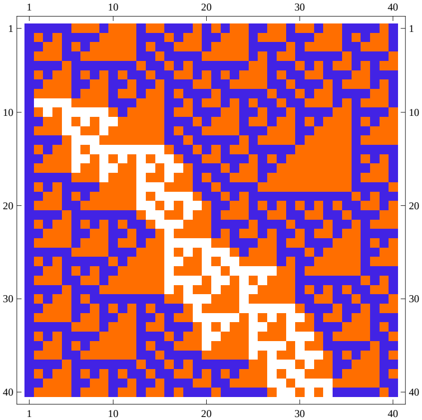

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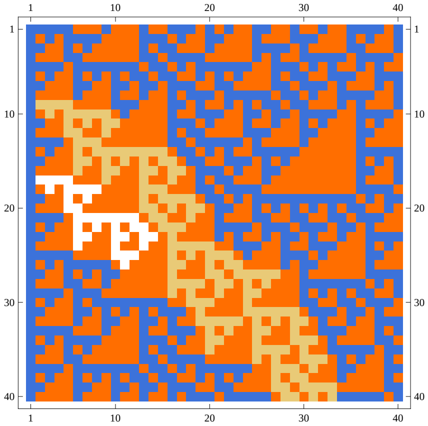

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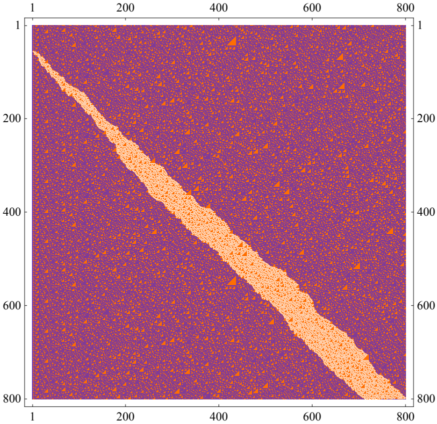

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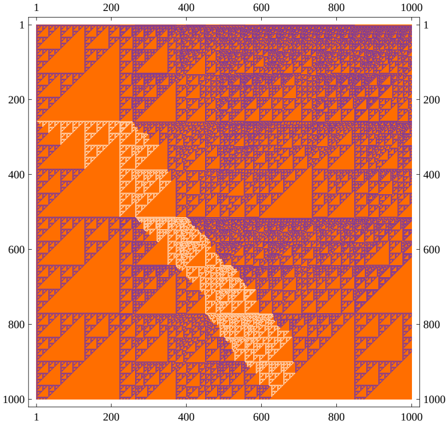

Figure : $p = 0.99$

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