

Funky CLTs for interacting particle systems.

(Joint with Remco van der Hofstad and Ed Perkins and others)

Nov. 2015

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(v.d.Hofstad, H. 2013)

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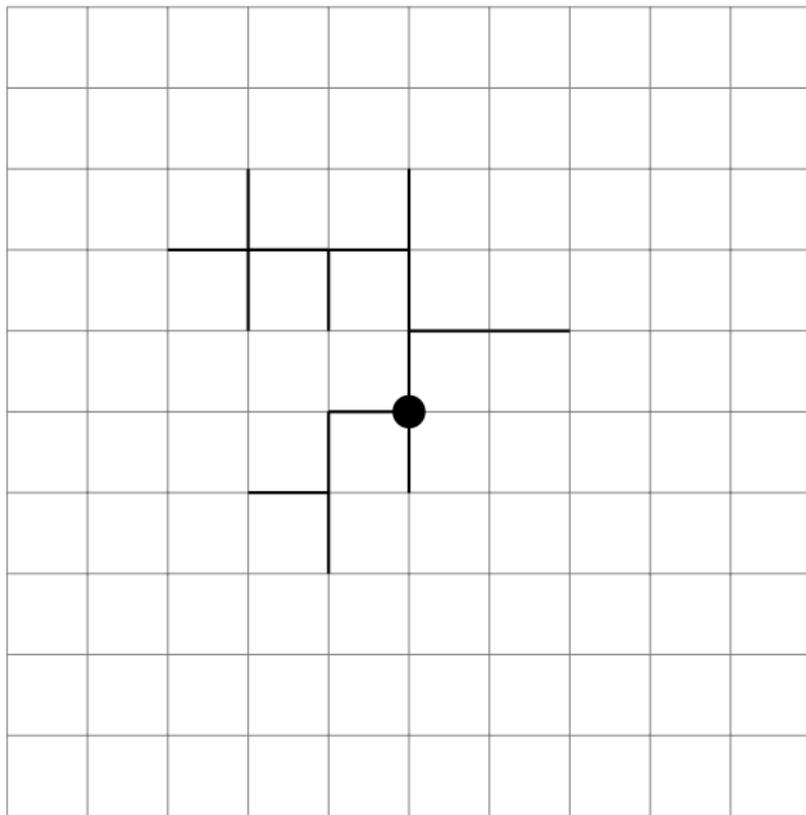
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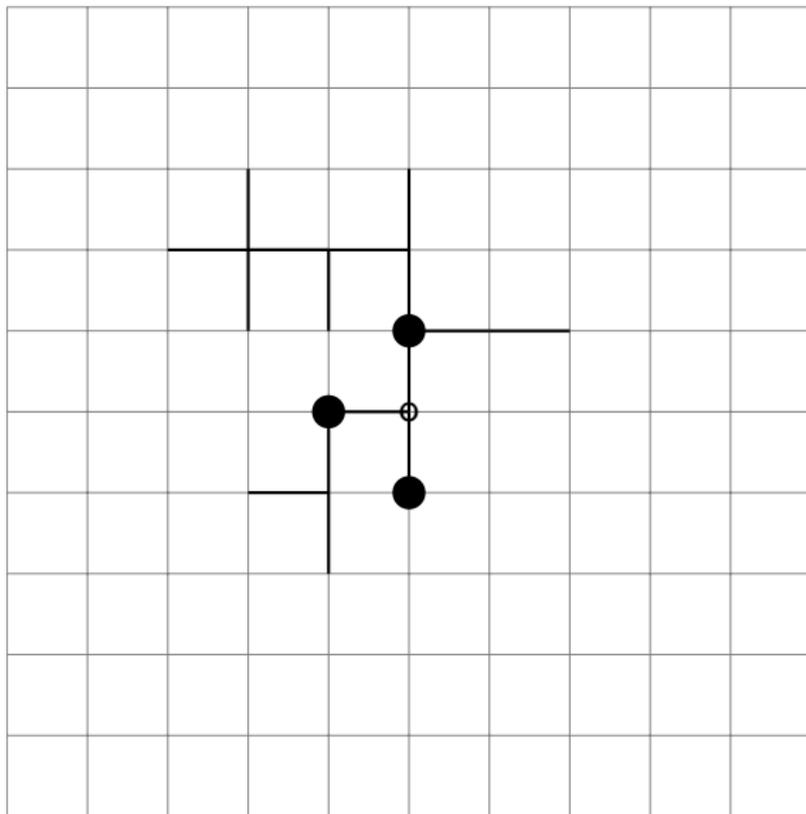
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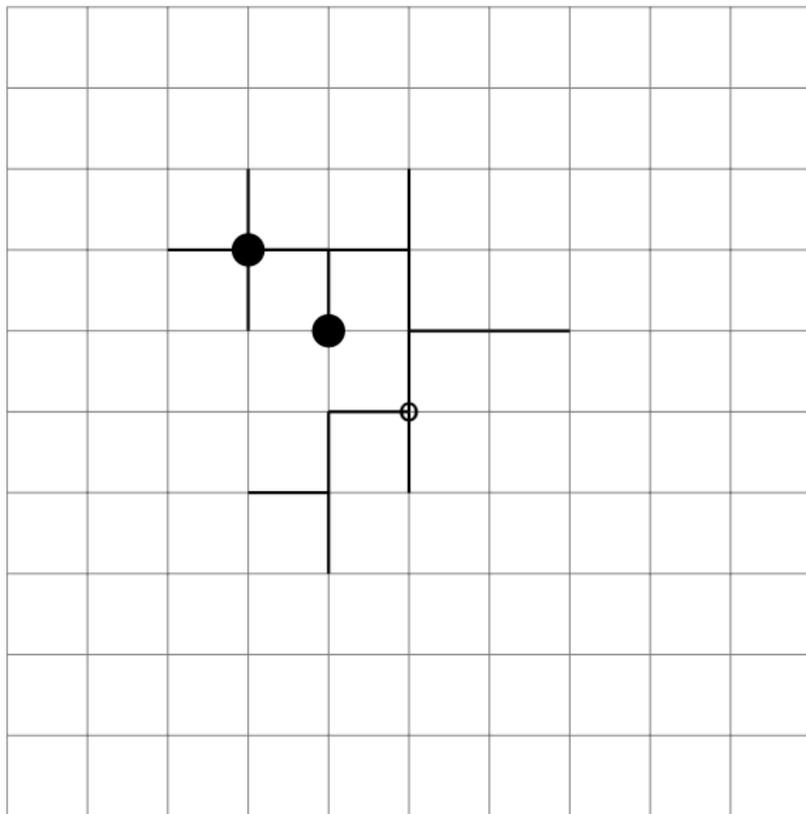
Lattice tree MVP0



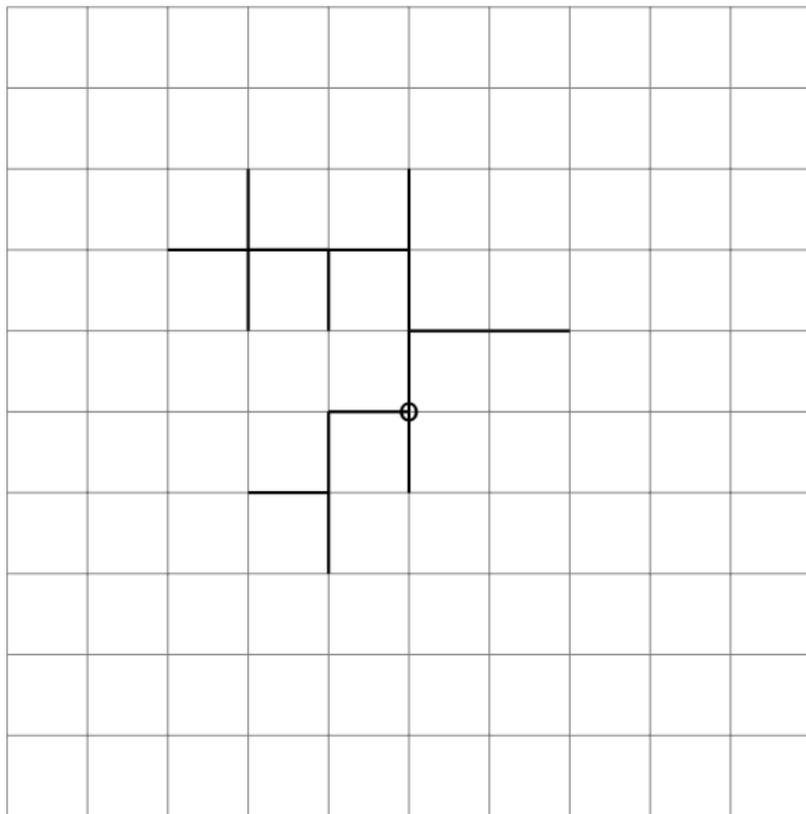
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Lattice tree MVP4



Lattice tree MVP6



Convergence of the measure-valued processes

For each d , there exists a measure \mathbb{N}_0 (CSBM) such that $\mu_n \xrightarrow{w} \mathbb{N}_0$ and for every $\varepsilon > 0$, $\mathbb{P}_n^\varepsilon \xrightarrow{w} \mathbb{P}_{\mathbb{N}_0}^\varepsilon$ for:

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- ▶ should also be true for (critical) oriented percolation and contact process with $d > 4$.

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In fact, not even \mathbb{N}_0 -measurable when $d < 4$.

A measure-on-paths-valued process

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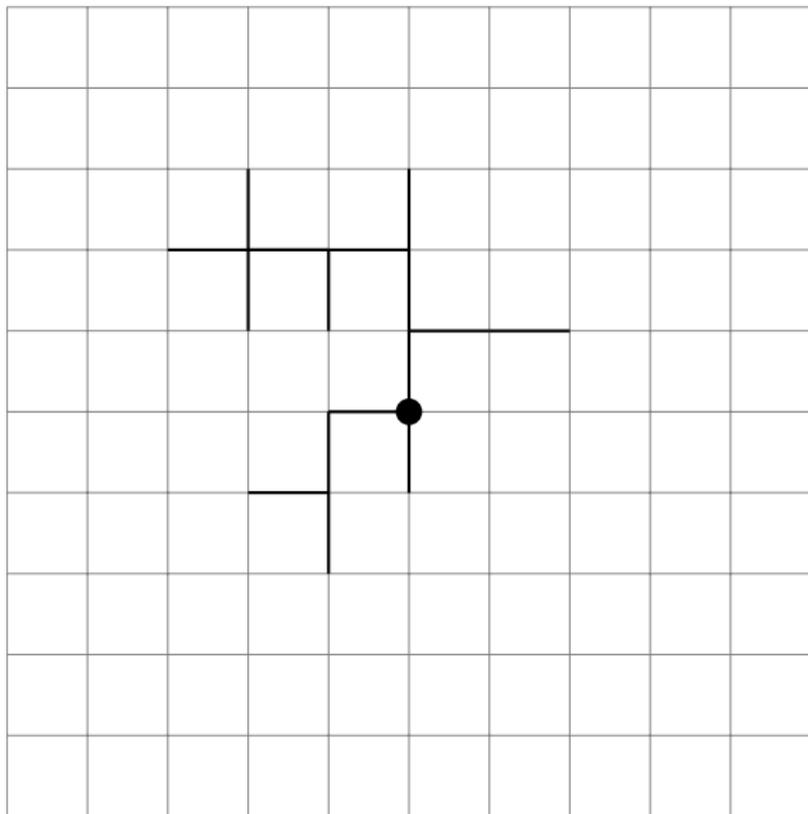
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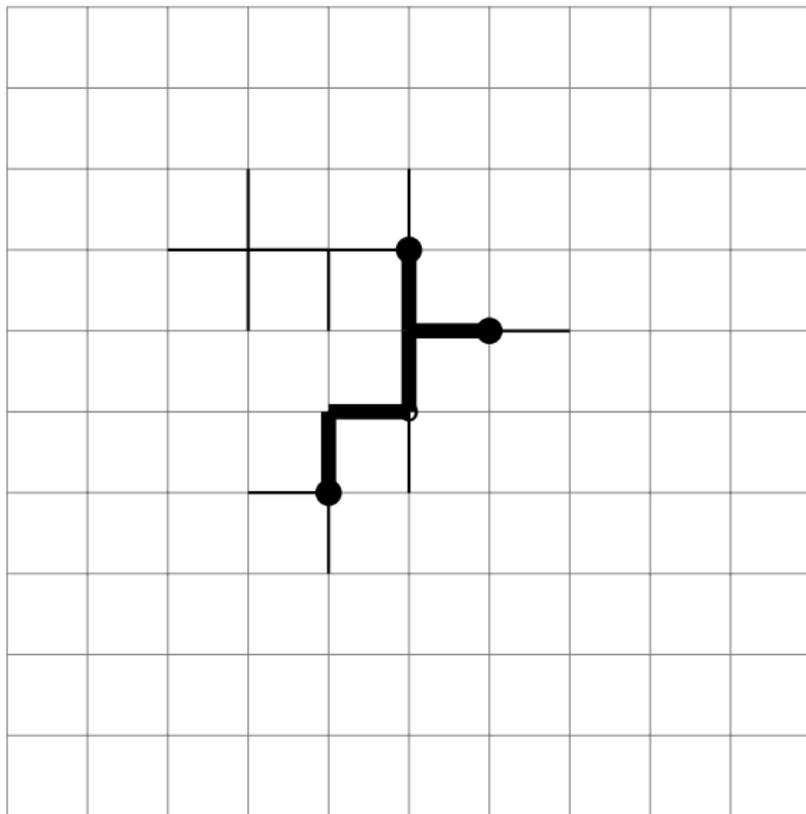
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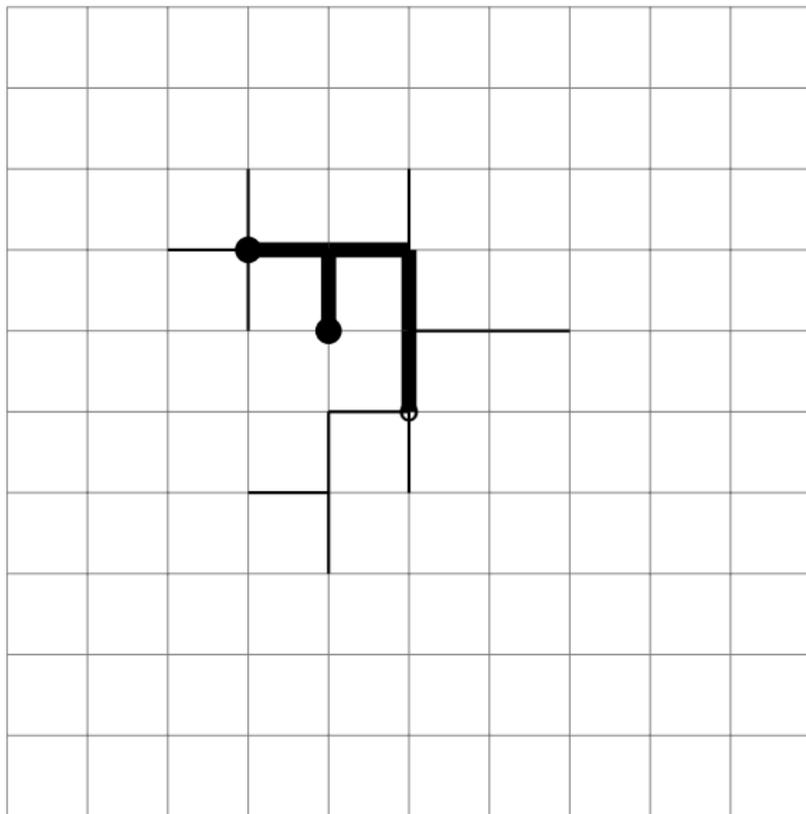
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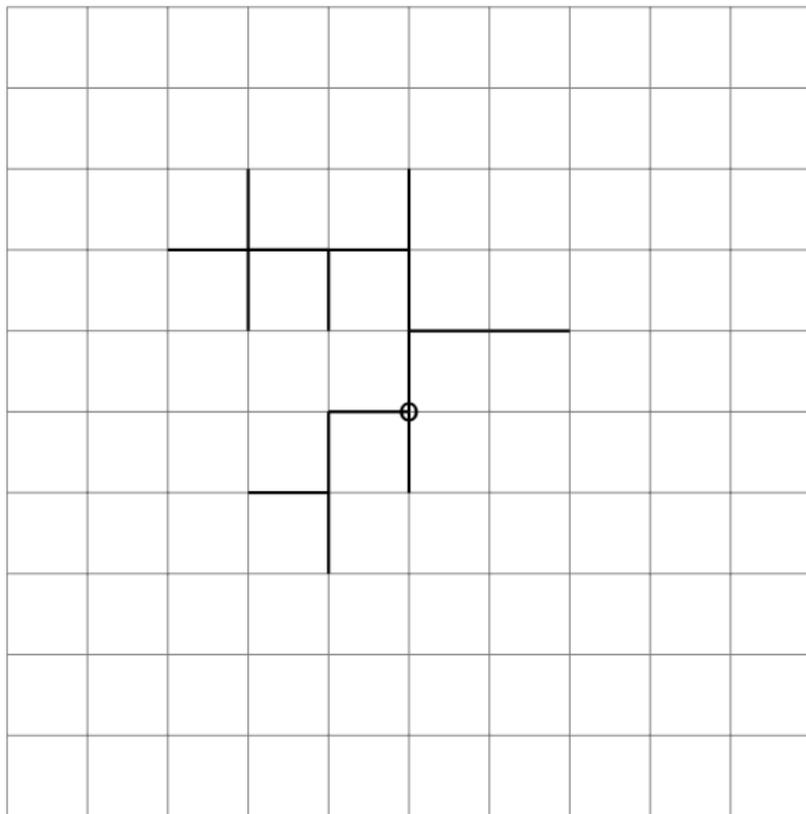
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Here $0 = s_0 < \dots < s_m$ and $\vec{k} \in \mathbb{R}^{dm}$ and

$$\phi_{\vec{s}, \vec{k}}(\mathbf{y}) \equiv \prod_{j=1}^m e^{i\mathbf{k}_j(\mathbf{y}(s_j) - \mathbf{y}(s_{j-1}))}, \quad (1)$$

and $\mathcal{G} = \{ \phi_{\vec{s}, \vec{k}} : \vec{s}, \vec{k} \text{ as above for some } m \in \mathbb{N} \}$.

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For voter model, (ii) is: CTRW \xrightarrow{w} BM.

Comments:

Let S_n be simple random walk. Then

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \mathbb{P} \left(S_{nt} = x, \left(\frac{S_{ns}}{\sqrt{n}} \right)_{s \leq t} \in \bullet \right) \\ &= \mathbb{P} \left(\left(\frac{S_{ns}}{\sqrt{n}} \right)_{s \leq t} \in \bullet \right) \rightarrow \mathbb{P} \left((B_s)_{s \leq t} \in \bullet \right). \end{aligned}$$

Now

$$\mathbb{E}_{\mu_n} [H_t(\bullet)] = C \sum_{x \in \mathbb{Z}^d} \mathbb{P} \left(x \in A_{nt}, \frac{\omega_{x,nt}}{\sqrt{cn}} \in \bullet \right).$$

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Convergence of the snakes to Brownian snake (Le Gall) would imply (*).

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$$\mathbb{E}_{\mu_n} [H_t(\phi_{\vec{s}, \vec{k}})] = C \sum_{\substack{\vec{x}: \\ x_0 = \mathbf{o}}} \left(\prod_{j=1}^{m+1} e^{i \frac{k_j}{\sqrt{n}} (x_j - x_{j-1})} \right) \mathbb{P} \left(\bigcap_{i=1}^{m+1} \left\{ (x_{i-1}, ns_{i-1}) \rightarrow (x_i, ns_i) \right\} \right),$$

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