On the Ginibre point process and its applications

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(Examples from joint works with P. Keeler, N. Ross, LHY Chen and A. Roellin)

A revisit of a Poisson process

Let $\{N_t\}_{t\geq 0}$ be a PP on $\mathbb{R}_+$

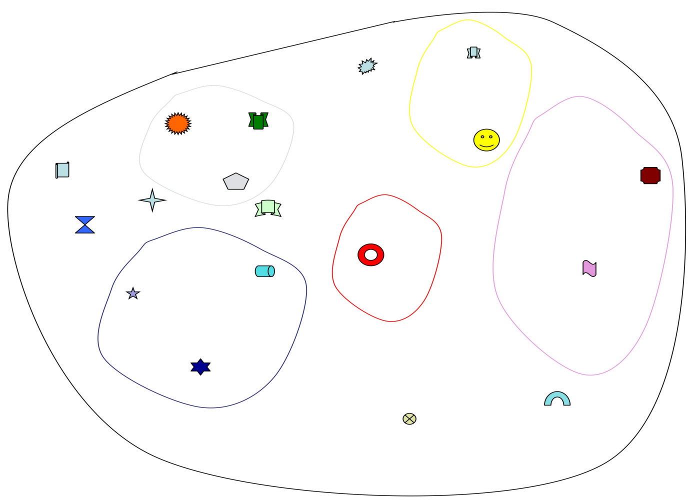
- It's determined by the arrival times $\{T_1, T_2, \ldots\}$
- $\{T_1, T_2, \ldots\}$ can be viewed as a random (countable) subset of $\mathbb{R}_+$
- For each Borel set $B \subset \mathbb{R}_+$, define $N \cap B := \{T_1, T_2, \ldots\} \cap B$, then $|N \cap B| = \#(N \cap B)$ counts the number of arrivals in B.
- $|N \cap B| \sim \text{Pn}(\lambda L(B))$, where $L(B)$ is the length of the set B.
- For $n \geq 2$, disjoint Borel sets $B_1, \ldots, B_n \subset \mathbb{R}_+$, $|N \cap B_1|, \ldots, |N \cap B_n|$ are independent.

Poisson point process on $\mathbb{R}^d$

The random countable subset Π of $\mathbb{R}^d$ is called a *Poisson process* with *mean measure* Λ if, for all $A \in \mathcal{B}(\mathbb{R}^d)$, the random variables $\Pi(A) := |\Pi \cap A|$ satisfy

(PP1) $\Pi(A) \sim \mathrm{Pn}(\Lambda(A))$,

(PP2) for all $n \geq 2$, if $A_1, \ldots, A_n$ are disjoint sets in $\mathcal{B}(\mathbb{R}^d)$, then $\Pi(A_1), \ldots, \Pi(A_n)$ are independent.



Palm processes (1)

- Assume Λ is diffuse, i.e., for each x, $\Lambda(\{x\}) = 0$.
- For r small, $\Pi(B(x,r))$ is roughly a Bernoulli rv with success prob $\Lambda(B(x,r))$ and

$$\begin{aligned}
\mathbb{E}[f(\Pi)\Pi(B(x,r))] &= \mathbb{E}[f(\Pi|_{B^c(x,r)} + \Pi|_{B(x,r)})\Pi(B(x,r))] \\
&\approx \mathbb{E}[f(\Pi|_{B^c(x,r)} + \delta_x)\Pi(B(x,r)) \\
&= \mathbb{E}[f(\Pi|_{B^c(x,r)} + \delta_x)]\mathbb{E}\Pi(B(x,r)) \\
&\approx \mathbb{E}[f(\Pi + \delta_x)]\mathbb{P}(\Pi(B(x,r)) = 1)
\end{aligned}$$

or

$$\mathbb{E}\{f(\Pi)|\Pi(B(x,r)) = 1\} \approx \mathbb{E}[f(\Pi + \delta_x)]$$

Palm processes (2)

- By letting $r \downarrow 0$, we obtain

$$\mathbb{E}\{f(\Pi)|\Pi(\{x\}) = 1\} = \mathbb{E}[f(\Pi + \delta_x)].$$

- The distribution of Π given that $\Pi(\{x\}) = 1$ is called the *Palm distribution* of Π at x.

- We say Π_x is the Palm process of Π at x if it has the Palm distribution of Π at x.

- $\Pi + \delta_x$ is a Palm process of Π at x.

The factorial moments

- The first factorial moment measure of Π is $\nu^{(1)}(dy) = \mathbb{E}\Pi(dy) = \Lambda(dy)$.
- For f on $\mathbb{R}^{2d}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^{2d}} f(y_1, y_2)\Pi(dy_1)\left(\Pi - \delta_{y_1}\right)(dy_2)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^{2d}} f(y_1, y_2)\Lambda(dy_1)\left(\Pi_{y_1} - \delta_{y_1}\right)(dy_2)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}^{2d}} f(y_1, y_2)\Lambda(dy_1)\Pi(dy_2)\right] \\
&= \int_{\mathbb{R}^{2d}} f(y_1, y_2)\Lambda^2(dy_1, dy_2),
\end{aligned}$$

so the second factorial moment measure of Π is $\nu^{(2)} = \Lambda^2$.

- In general, the nth factorial moment measure of Π is

$$\nu^{(n)} = \Lambda^n$$

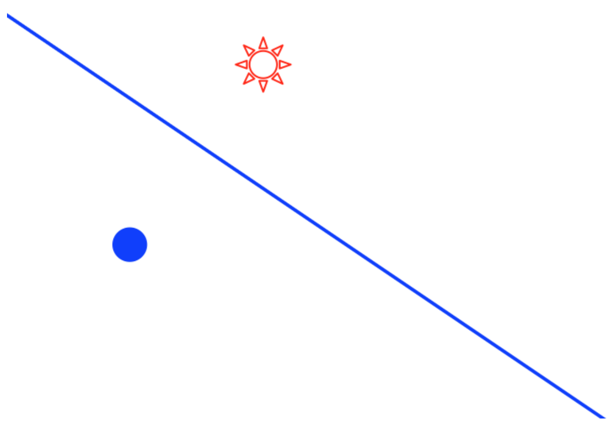
Point processes

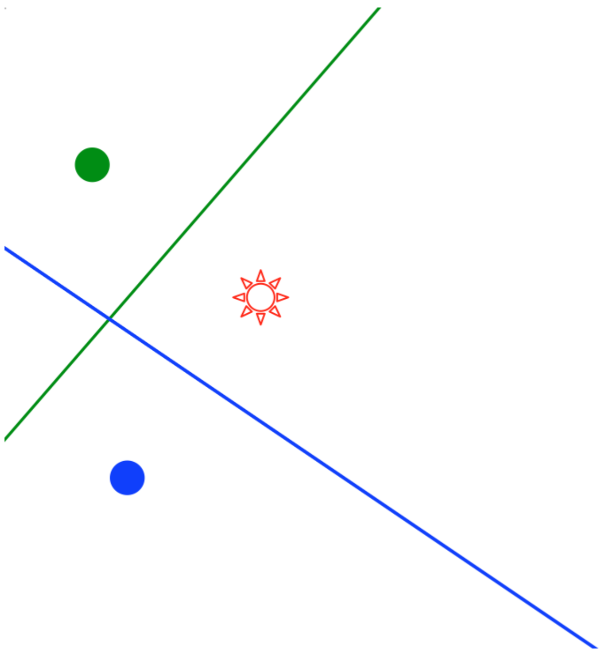
- In general, we can define a point process Ξ on a very general carrier space S, its Palm processes $\Xi_x \stackrel{\mathrm{d}}{=} \mathscr{L}(\Xi|\Xi(\{x\})=1),\ x\in S$, and its factorial moment $\nu^{(n)}$ for $n\ge 1$ if exists:

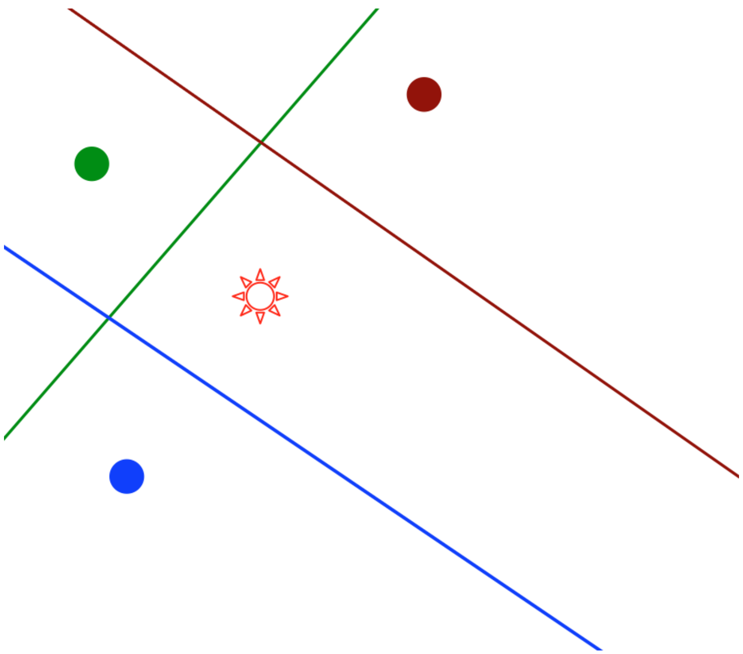
$$\nu^{(n)}(dy_1,\dots,dy_n) := \mathbb{E}\left\{\Xi(dy_1)\left(\Xi-\delta_{y_1}\right)(dy_2)\dots\left(\Xi-\sum_{i=1}^{n-1}\delta_{y_i}\right)(dy_n)\right\}.$$

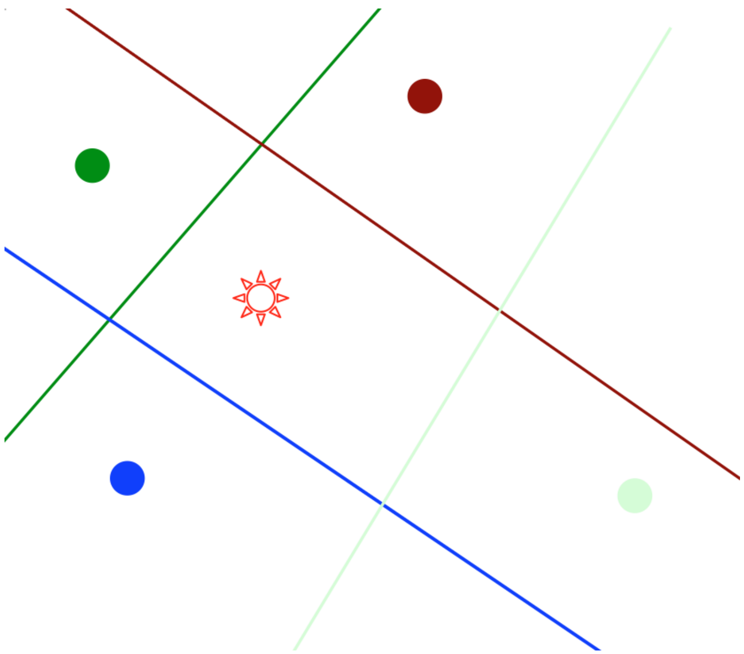
- The distribution of a point process Ξ can be specified through
  - its Palm processes;
  - its factorial moments $\nu^{(n)}$ for all $n \geq 1$ with some mild conditions on the factorial moments;
  - Janossy densities, compensators, etc.

Voronoi tessellation (Georgy Voronoy 1908)









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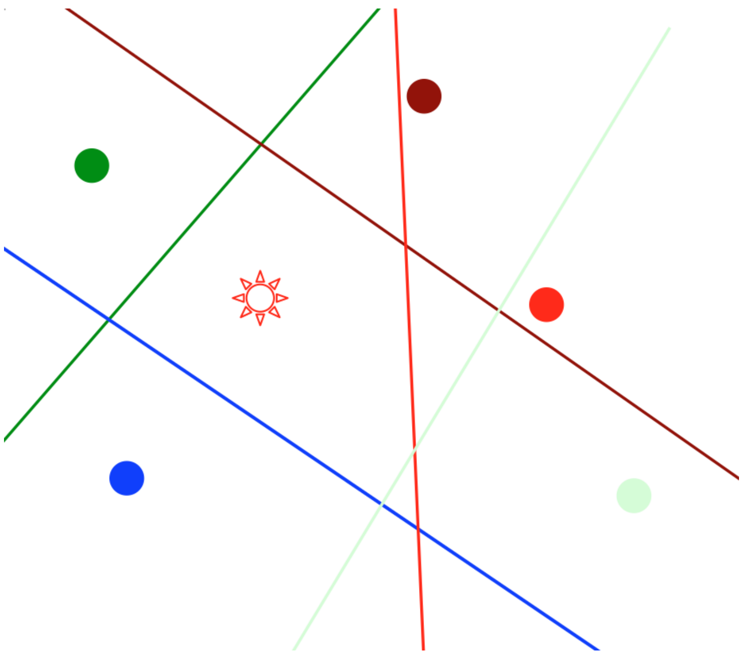

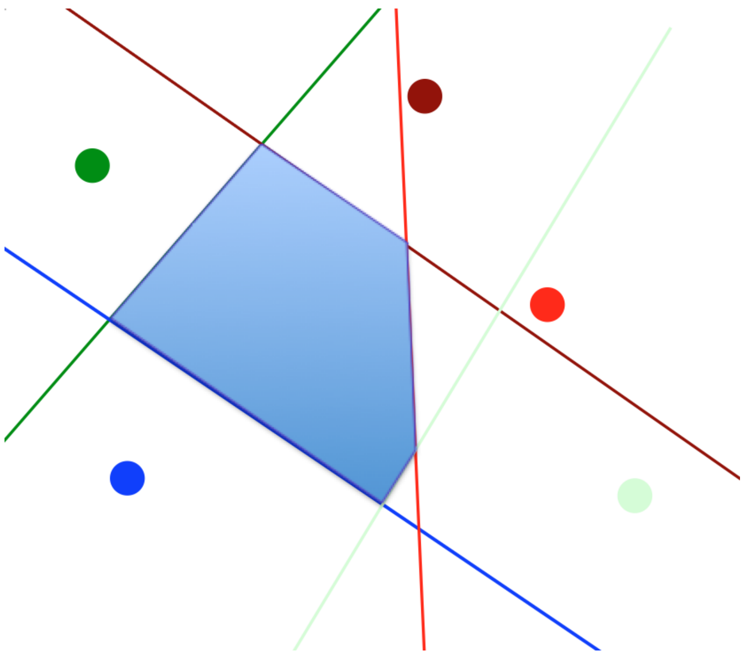

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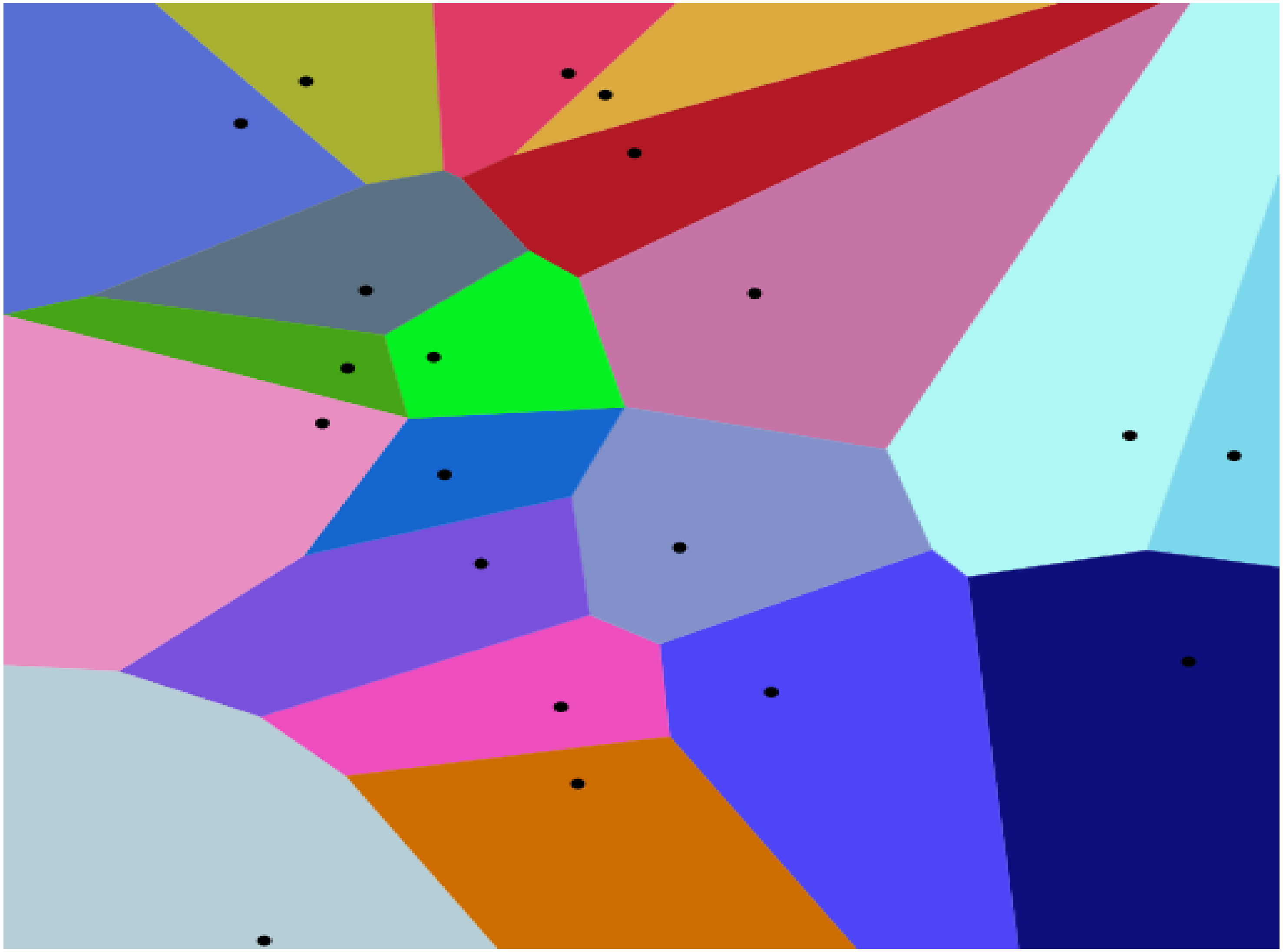


The deficiency of Poisson point process

- Signal transmitters for wireless networks: when a transmitter is built, it is unlikely that another one is built next to it (Błaszczyszyn and Yogeshwaran, 2014; Miyoshi and Shirai, 2014a,b; Torrisi and Leonardi (2014))
- The structure of the cells of biological tissues don't resemble the Voronoi tessellation associated to Poisson points (Le Caer and Ho 1990).

The Ginibre point process

We say the point process Ξ on the complex plane $\mathbb{C}$ ($\cong \mathbb{R}^2$) is the *Ginibre point process* if its factorial moment measures are given by

$$\nu^{(n)}(dx_1, \ldots, dx_n) = \rho^{(n)}(x_1, \ldots, x_n)dx_1 \ldots dx_n, \; n \geq 1,$$

where $\rho^{(n)}(x_1, \ldots, x_n)$ is the determinant of the $n \times n$ matrix with (i, j)th entry

$$K(x_i, x_j) = \frac{1}{\pi} e^{-\frac{1}{2}(|x_i|^2+|x_j|^2)} e^{x_i \bar{x}_j},$$

where $\bar{x}$ and $|x|$ are the complex conjugate and modulus of x.

Properties

- $\rho^{(1)} = \frac{1}{\pi}$, $\rho^{(2)}(y_1, y_2) = \frac{1-e^{-|y_1-y_2|^2}}{\pi^2}$, etc
- The Palm process Ξ_x satisfies (see Goldman 2010)

$$\Xi \stackrel{d}{=} (\Xi_x \setminus \{x\}) \cup \{x + Z\},$$

where $Z = (Z_1, Z_2\sqrt{-1})$ with (Z_1, Z_2) having bivariate normal on $\mathbb{R}^2$ with mean $(0, 0)$ and covariance matrix $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

  - The Palm process Ξ_x can be obtained by removing a point from the process which is Gaussian distributed from x and then adding x to Ξ.
  - It is still an open problem to know how Z is correlated with $\Xi_x \setminus \{x\}$ except $x + Z \notin \Xi_x \setminus \{x\}$.

α-Ginibre point process

The α-Ginibre point process can be obtained by deleting, independently and with probability $1 - \alpha$, each point of the Ginibre process and then applying the homothety of ratio $\sqrt{\alpha}$ to the remaining points to restore the intensity of the process.

- When $\alpha \to 0$, the α-Ginibre point process converges to Poisson process.

Signal transmitters for wireless networks

- Signal transmitters are located in a fashion very close to a realisation of an α-Ginibre point process $\{x_i\} \subset \mathbb{C}\backslash\{0\}$.

- Signals lose strength according to a loss function:

$$\ell : \ \mathbb{C}\backslash\{0\} \to (0, \infty).$$

- An iid shadowing sequence

$$0 < S_1, S_2, \ldots$$

- The power from transmitter x_i is $P_i = \ell(x_i)S_i > 0$, $i \geq 1$ and the signal power point process is

$$\{P_i\} \subset (0, \infty).$$

- The signal power process $\{P_i\}$ has many very weak signals near zero.
- (Keeler, Ross, X.) For an interval $[a, b] \subset (0, \infty)$, the signal power process with strengths in $[a, b]$ is

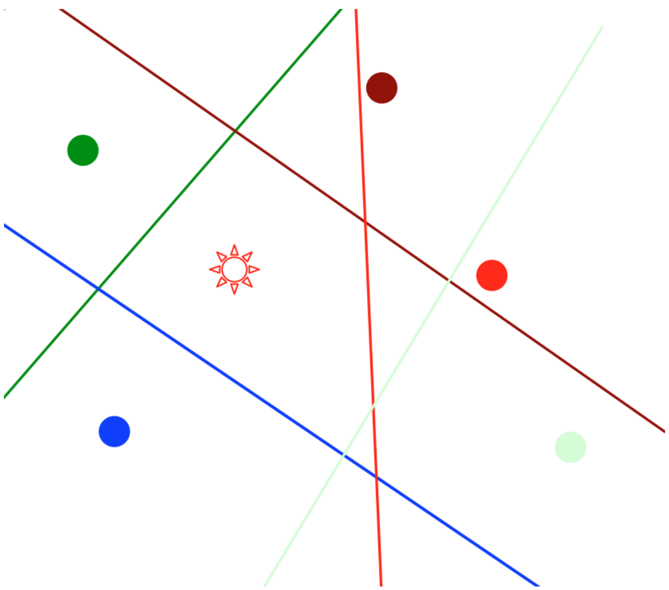
$$\{P_i\} \cap [a, b]$$

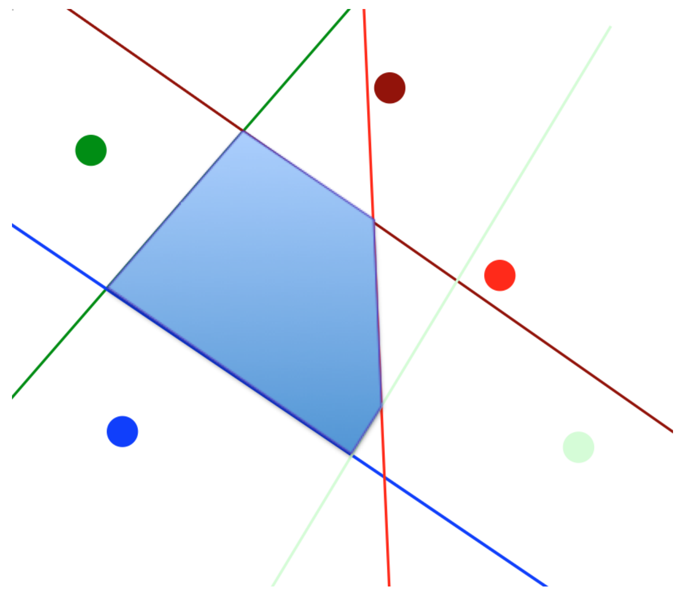
and $\{P_i\} \cap [a, b]$ is approximately a Poisson process under some mild conditions.

- The proof is based on coupling plus something else.

The Ginibre tessellation

- We consider the window $Q_\lambda := \{(s_1, s_2\sqrt{-1}) : \ -0.5\sqrt{\lambda} \le s_1, s_2 \le 0.5\sqrt{\lambda}\} \subset \mathbb{C}$ (Schreiber and Yukich (2013)).
- For a realization $\boldsymbol{x}$ of Ξ and $x \in \boldsymbol{x}$, let $\mathcal{C}(x, \boldsymbol{x})$ be the set of every point in $\mathbb{C}$ whose (Euclidean) distance to x is less than or equal to its distance to any other point of $\boldsymbol{x}$.





- The set $\mathcal{C}(x, \boldsymbol{x})$ is called *the Voronoi cell* centred at x and the collection of $\mathcal{C}(x, \boldsymbol{x})$, $x \in \boldsymbol{x}$, is called the *Voronoi tessellation* induced by $\boldsymbol{x}$.

- We define the random measure

$$\boldsymbol{X}(dx) = L(x, \Xi)\Xi(dx),$$

where $L(x, \Xi) := L_\lambda(x, \Xi)$ is one half the total edge length of the *finite* length edges (hence we exclude all infinite edges) in the cell $\mathcal{C}(x, \boldsymbol{X})$.

- The total edge length of the Ginibre-Voronoi tessellation induced by Ξ with centers in $\Xi \cap Q_\lambda$ can be written as

$$\mathcal{L}(\lambda) := |\boldsymbol{X}| = \int_{Q_\lambda} L(x, \Xi)\Xi(dx).$$

Theorem (Chen, Roellin and X)

Let $B^2 = \mathrm{Var}(\mathcal{L}(\lambda))$ and $W = \frac{\mathcal{L}(\lambda) - \mathbb{E}\mathcal{L}(\lambda)}{B}$. We have

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\mathcal{L}(\lambda) \in (0,\infty), \qquad \lim_{\lambda\to\infty} \lambda^{-1}B^2 \in (0,\infty)$$

and

$$d_{\mathrm{K}}(\mathcal{L}(W), \mathcal{N}(0,1)) = O\left(\lambda^{-1/2}\ln\lambda\right).$$

Remark The Ginibre-Voronoi tessellation is a special case of the Gibbs Voronoi tessellations studied in Xia and Yukich (2015):

$$d_{\mathrm{K}}(\mathcal{L}(W), \mathcal{N}(0,1)) = O\left(\lambda^{-1/2}(\ln\lambda)^4\right).$$

Idea of the proof

- There is a local dependence structure for the Ginibre-Voronoi tessellation.
- Stein's method can be employed to deal with the local dependence.
- A 'miracle' happens after some messy algebraic manipulation using the Palm processes of $\boldsymbol{X}$ and the coupling techniques.
- The same approach can sharpen other statistics of random geometric and related Euclidean graphs on the Ginibre point process.

Thank you!