Generalizing L

W. Hugh Woodin

Harvard University

January 11, 2017

Generalizing the projective sets

Recall: the projective sets

Definition

A set $A \subseteq \mathbb{R}^n$ is **projective** if it can be generated from the open subsets of $\mathbb{R}^n$ in finitely many steps of taking complements and images by continuous functions,

$$f : \mathbb{R}^n \to \mathbb{R}^n.$$

Definition

Suppose that $A \subseteq \mathbb{R} \times \mathbb{R}$. A function f **uniformizes** A if for all $x \in \mathbb{R}$:

- if there exists $y \in \mathbb{R}$ such that $(x, y) \in A$ then $(x, f(x)) \in A$.

Two questions of Luzin

Two questions of Luzin

1. *Suppose* $A \subseteq \mathbb{R} \times \mathbb{R}$ *is projective. Can* A *be uniformized by a projective function?*
2. *Suppose* $A \subseteq \mathbb{R}$ *is projective. Is* A *Lebesgue measurable and does* A *have the property of Baire?*

Both questions are unsolvable on the basis of the ZFC **axioms**

Projective Determinacy (PD) and the Luzin's questions

Definition

Projective Determinacy (PD): Every projective set $A \subseteq \mathbb{R}$ is determined.

Theorem

Assume every projective set is determined.

(1) (Mycielski-Steinhaus) *Every projective set has the property of Baire.*

(2) (Mycielski-Swierczkowski) *Every projective set is Lebesgue measurable.*

(3) (Moschovakis) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

The axiom $V = L$ and the projective sets

Theorem

Assume $V = L$.

(1) (Gödel) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

(2) (Gödel) *There is a projective set which is not Lebesgue measurable:*

- *there is a projective wellordering of the reals.*

So again we see:

(meta) Corollary

$V \neq L$.

A transfinite extension of the projective sets

Relativizing L to $\mathbb{R}$

Define $L_\alpha(\mathbb{R})$ by induction on α by:

1. $L_0(\mathbb{R}) = \mathbb{R}$ (more precisely $L_0(\mathbb{R}) = V_{\omega+1}$),
2. (Successor case) $L_{\alpha+1}(\mathbb{R}) = \mathcal{P}_{\mathrm{Def}}(L_\alpha(\mathbb{R}))$,
3. (Limit case) $L_\alpha(\mathbb{R}) = \cup\{L_\beta(\mathbb{R}) \mid \beta < \alpha\}$.

$L(\mathbb{R})$ is the class of all sets X such that $X \in L_\alpha(\mathbb{R})$ for some ordinal α.

- $L_1(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *is exactly the projective sets.*
- $L_{\omega_1}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *is the smallest* σ*-algebra containing the projective sets and closed under images by continuous functions,* $f : \mathbb{R} \to \mathbb{R}$.

A transfinite extension of projective determinacy

The axiom $L(\mathbb{R}) \models \mathrm{AD}$

Suppose $A \subseteq \mathbb{R}$ and $A \in L(\mathbb{R})$. Then A is determined.

Theorem (Martin-Steel, Woodin)

Assume there are infinitely many Woodin cardinals with a measurable cardinal above. Then $L(\mathbb{R}) \models \mathrm{AD}$.

Theorem

The following theories are equiconsistent.

(1) $\mathrm{ZFC} +$ "$L(\mathbb{R}) \models \mathrm{AD}$".

(2) $\mathrm{ZFC} +$ "There are infinitely many Woodin cardinals".

Luzin's questions beyond the projective sets

Luzin's questions at stage $L_\alpha(\mathbb{R})$

1. *Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ and $A \in L_\alpha(\mathbb{R})$. Can A be uniformized by a function $f \in L_\alpha(\mathbb{R})$?*
2. *Suppose $A \subseteq \mathbb{R}$ and $A \in L_\alpha(\mathbb{R})$. Is A Lebesgue measurable and does A have the property of Baire?*

Assume $L(\mathbb{R}) \models \mathrm{AD}$.

1. *The answer to Luzin's measure question is yes at every ordinal stage $L_\alpha(\mathbb{R})$.*
2. *The answer to Luzin's uniformization question is yes at many ordinal stages $L_\alpha(\mathbb{R})$,*
    - *but not for all α.*

Uniformization in $L(\mathbb{R})$

Lemma

The following are equivalent.

(1) *Uniformization holds for $L(\mathbb{R})$.*

(2) *For all sufficiently large ordinals α, uniformization holds for $L_\alpha(\mathbb{R})$.*

(3) *The Axiom of Choice holds in $L(\mathbb{R})$.*

Uniformization in $L(\mathbb{R})$ and the necessity of determinacy

Question

Can positive answers to Luzin's questions be obtained by a different method not involving determinacy?

Theorem

Suppose that uniformization holds in $L_\alpha(\mathbb{R})$ and that $\alpha = \omega_1 \cdot \beta$ for some limit ordinal β. Then the following are equivalent.

(1) *Every set $A \in L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is Lebesgue measurable and has the property of Baire.*

(2) *Every set $A \in L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is determined.*

- The restriction on α is necessary:
  - the theorem is **false** with $\alpha = \omega_1$.

The ultimate generalization of the projective sets

Universally Baire sets

Speculation

The enlargements of L we seek will also yield enlargements of $L(\mathbb{R})$.

- *In seeking the ultimate enlargement of L perhaps we should first seek the ultimate enlargement of $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.*

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is **universally** Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}^n$, the preimage of A by π has the property of Baire in the space Ω.

- Universally Baire sets are necessarily Lebesgue measurable and have the property of Baire.

$L(A, \mathbb{R})$ where $A \subseteq \mathbb{R}$

Relativizing L to $A \subseteq \mathbb{R}$

Suppose $A \subseteq \mathbb{R}$. Define $L_\alpha(A, \mathbb{R})$ by induction on α by:

1. $L_0(A, \mathbb{R}) = \mathbb{R} \cup \{A\}$ (more precisely $L_0(A, \mathbb{R}) = V_{\omega+1} \cup \{A\}$),
2. (Successor case) $L_{\alpha+1}(A, \mathbb{R}) = \mathcal{P}_{\mathrm{Def}}(L_\alpha(A, \mathbb{R}))$,
3. (Limit case) $L_\alpha(A, \mathbb{R}) = \cup\{L_\beta(A, \mathbb{R}) \mid \beta < \alpha\}$.

- $L(A, \mathbb{R})$ is the class of all sets X such that $X \in L_\alpha(A, \mathbb{R})$ for some ordinal α.

- $\mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(A, \mathbb{R})$ is the smallest σ-algebra containing A and closed under images by continuous functions $f : \mathbb{R} \to \mathbb{R}$.
- If $A \in L(\mathbb{R})$ then $L(A, \mathbb{R}) = L(\mathbb{R})$.

An abstract generalization of the projective sets

Theorem

Suppose that there is a proper class of Woodin cardinals. Then every projective set is universally Baire.

Theorem

Suppose that there is a proper class of Woodin cardinals.

(1) (Martin-Steel) *Suppose* $A \subseteq \mathbb{R}$ *is universally Baire. Then* A *is determined.*

(2) *Suppose* $A \subseteq \mathbb{R}$ *is universally Baire. Then every set* $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *is universally Baire.*

(3) (Steel) *Suppose* $A \subseteq \mathbb{R} \times \mathbb{R}$ *is universally Baire. Then* A *can be uniformized by a universally Baire function.*

- The answers to Luzin's questions are both yes for the universally Baire sets.

Determinacy and the universally Baire sets

Theorem (Martin-Steel, Woodin)

Assume there are infinitely many Woodin cardinals with a measurable cardinal above. Then

- $L(\mathbb{R}) \models \mathrm{AD}$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that $A \subset \mathbb{R}$ is universally Baire. Then

- $L(A, \mathbb{R}) \models \mathrm{AD}$.

Measuring the complexity of universally Baire sets

Definition

Suppose A and B are subsets of $\mathbb{R}$.

1. A is **borel reducible** to B, $A \leq_{\text{borel}} B$, if there is a borel function $\pi : \mathbb{R} \to \mathbb{R}$ such that
   - either $A = \pi^{-1}[B]$ or $A = \mathbb{R} \setminus \pi^{-1}[B]$.
2. A and B are **borel bi-reducible** if
   - $A \leq_{\text{borel}} B$ and $B \leq_{\text{borel}} A$.
3. The **borel degree** of A is the equivalence class of all sets which are borel bi-reducible with A.

Theorem (Martin-Steel, Martin, Wadge)

Assume there is a proper class of Woodin cardinals.

Then the borel degrees of the universally Baire sets are linearly ordered by borel reducibility and moreover this is a wellorder.

Speculation

Perhaps this ultimate generalization of the projective sets can lead us to the ultimate generalization of L.

- But how?

Weak extender models and universality

Definition

Suppose λ is an uncountable cardinal.

- λ is a **singular cardinal** if there exists a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.
- λ is a **regular cardinal** if there does not exist a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.

Lemma (Axiom of Choice)

Every (infinite) successor cardinal is a regular cardinal.

Definition

Suppose λ is an uncountable cardinal. Then $\mathrm{cof}(\lambda)$ is the minimum possible $|X|$ where $X \subset \lambda$ is cofinal in λ.

- $\mathrm{cof}(\lambda)$ is always a regular cardinal.
- If λ is regular then $\mathrm{cof}(\lambda) = \lambda$.
- If λ is singular then $\mathrm{cof}(\lambda) < \lambda$.

Supercompactness

Definition

Suppose that κ is an uncountable regular cardinal and that $\kappa < \lambda$.

1. $\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \,\big|\, |\sigma| < \kappa\}$.
2. Suppose that $U \subseteq \mathcal{P}\left(\mathcal{P}_\kappa(\lambda)\right)$ is an ultrafilter.
   - U is **fine** if for each $\alpha < \lambda$,
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \,\big|\, \alpha \in \sigma\} \in U.$$
   - U is **normal** if for each function
     $$f : \mathcal{P}_\kappa(\lambda) \to \lambda$$
     such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \,\big|\, f(\sigma) \in \sigma\} \in U,$$
     there exists $\alpha < \lambda$ such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \,\big|\, f(\sigma) = \alpha\} \in U.$$

Definition

Suppose that κ is an uncountable regular cardinal. Then κ is a **supercompact cardinal** if for each $\lambda > \kappa$ there exists an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that:

1. U is κ-complete,
2. U is a normal fine ultrafilter.

Lemma

Suppose κ is an uncountable regular cardinal. Then the following are equivalent.

(1) *κ is a supercompact cardinal.*

(2) *For each $\lambda > \kappa$, there exists an elementary embedding*

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, and such that $M^\lambda \subset M$.

- One can require that the transitive class M and the embedding j each be Σ_2-definable in V from parameters.

Weak Extender Models

Definition

A transitive class N model of ZFC is a **weak extender model for δ is supercompact** iff for every $\gamma > \delta$ there exists a δ-complete normal fine measure U on $\mathcal{P}_\delta(\gamma)$ such that

1. $N \cap \mathcal{P}_\delta(\gamma) \in U$,
2. $U \cap N \in N$.

The Basic Thesis

If there is a generalization of L at the level of a supercompact cardinal then it should exist in a version which is a weak extender model for the supercompactness of some δ.