Vaught's conjecture.

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Summary of the talk

- Part I: Vaught's conjecture.
- Part II: Vaught's conjecture in Model Theory.
- Part III: Vaught's conjecture in Computability Theory.
- Part IV: Vaught's conjecture in Descriptive Set Theory.

Part I: Vaught's Conjecture

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A has *continuum many elements* if $|A| = |\mathbb{R}|$.

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Thm [Silver 1980]: If $\equiv$ is a Borel equivalence relation on $\mathbb{R}$, then the number of equivalence classes is either countable or continuum.

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Thm:[Cohen] Consistently with ZFC,

$2^{\aleph_0}$ can be any cardinal κ so long as $|\kappa^{\aleph_0}| = \kappa$.

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- The ordered ring $(\mathbb{Z}; 0, 1, +, \times, \leq)$.

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1. $\forall x, y, z(x +_K (y +_K z) = (x +_K y) +_K z)$
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3. $\forall x(x +_K 0_K = x)$
4. $\forall x((\exists y)x +_K y = 0_K)$
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6. $\forall x(x \times_K 1_K = x)$
7. $\forall x(x \neq 0_K \to ((\exists y)x \times_K y = 1_K))$
8. $\forall x, y, z(x \times_K (y +_K z) = (x \times_K y) +_K (x \times_K z))$
9. $\forall x(x \neq 0_K \to (x <_K 0 \vee 0_K < x))$
10. $\forall x, y(x <_K y \to \forall z(z +_K x < z +_K y))$
11. $\forall x, y(x <_K y \to \forall z(z > 0_K \to (z \times_K x < z \times_K y)))$

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Throughout this talk, vocabularies are always countable.

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Examples:

- The class of groups
- The class of rings
- The class of commutative rings with no zero-divisors
- The class of linear orderings
- The class of dense linear orderings without end-points
- The class of algebraically closed fields
- The class of $\mathbb{Q}$-vector spaces.

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Note: Some of these possibilities might end up being isomorphic, and hence are being counted multiple times.

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The most important partial result is:

Theorem: [Morley 70]
The number of countable models on any axiomatizable class is either countable, $\aleph_1$, or $2^{\aleph_0}$.

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Then, by Silver's theorem, every axiomatizable class of structures has either countably many or continuum many elementary equivalence classes.

Types

Let $\mathcal{A}$, $\mathcal{B}$ be structures, $\bar{a} \in \mathcal{A}^{<\omega}$, and $\bar{b} \in \mathcal{B}^{<\omega}$.

Definition: $\bar{a}$ and $\bar{b}$ have the *same type* if, for every sentence $\varphi(\bar{x})$,

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Proof: Type equivalence is Borel. By Silver's theorem, every axiomatizable class of structures has either countably many or continuum many different types.

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Definition $\bar{a}$ and $\bar{b}$ have the *same type* if they satisfy the same formulas.

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Since there are countably or continuum many elementary equivalence classes,
Martin's conjecture $\implies$ Vaught's conjecture.

Case study: ω-Stable Theories

Definition: A theory is *ω-stable* if, in every model, even after naming countably many elements, there are countably many types.

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Theorem: [Shelah, Bouscaren] ω-stable theories satisfy Martin's conjecture.

Summary of the talk

- Part I: Vaught's conjecture.
- Part II: Vaught's conjecture in Model Theory.
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From a computability viewpoint,

a counterexample to Vaught's conjecture

must look like the class of ordinals.

Computable functions

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Recall that any finite object can be encoded by a natural number.

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The Halting problem: The set of programs that **halt**, and don't run for ever, is **not** computable.

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Def: For $A, B \subseteq \mathbb{N}$, *A is computable in B*, written $A \leq_T B$,
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Which is $<_T$ the following set:

- The set of true sentences in number theory.

Computable Mathematics

Study

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Various areas have been studied,

1. Combinatorics,
2. Algebra,
3. Analysis,
4. Model Theory

In many cases one needs to develop a better understanding of the mathematical structures to be able to get the computable analysis.

Coding structures

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Note: A single structure can have many isomorphic presentations.

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Recall: A countable ring $\mathcal{A} = (A, 0, 1, +_A, \times_A)$ can be encoded by three sets $A \subseteq \mathbb{N}$, $+_A \subseteq \mathbb{N}^3$ and $\times_A \subseteq \mathbb{N}^3$.

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However, maximal ideals can be found computable in the halting problem.
There are computable rings, all whose maximal ideals compute the halting problem.

Example: Represent Structures

Def: A group $\mathcal{G} = (G, +)$ is *computable*
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If X is the complement of the halting problem, then any presentation of $\mathcal{G}_X$ computes the halting problem.

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Examples:

- For a ring $\mathcal{A}$, the $\mathcal{A}[x]$ is computable in $\mathcal{A}$.
- For every group $\mathcal{G}$, $\mathcal{G}$ is computable in $\mathcal{G} \oplus \mathcal{G}$ and vice-versa.
- There are groups $\mathcal{G}$ such that $\sum_{i=1}^{\infty} \mathcal{G}$ does not compute $\mathcal{G}$.
  Take $G = \sum_{i=1}^{\infty} (\mathbb{Z}_{p_i})^{k_i}$, non-computable with all $k_i \neq 0$.

Vaught's conjecture in Computability Theory

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Proof of (2) $\implies$ (1):

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Proof of (2) $\implies$ (1):
Lemma: If $(\mathcal{L}, \leq)$ is a linear ordering such that every element has at most countably many elements below it, then it has at most $\aleph_1$ many elements.

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However, the class of well-orders is not 1st-order axiomatizable.

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   X and Y compute the same ordinals
   if and only if
   X and Y compute the same structures in $\mathbb{K}$.

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Definition: A topological space $\mathcal{X}$ is *Polish* if
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Definition: The *orbit* of an element $x \in \mathcal{X}$ is

$$G \cdot x = \{g \cdot x : \quad g \in G\}.$$

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Borel sets

Definition: The *Borel sets* of a topological space $\mathcal{X}$ is the smallest class of subsets of $\mathcal{X}$ that

- contains all open sets
- and is closed under
  - countable unions;
  - countable intersections;
  - complements.

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Theorem:
The topological Vaught's conjecture holds for the following groups:

- Locally compact
- Abelian [Sami]
- Nilpotent [Hjorth and Solecki]
- Groups with two-sided invariant metrics [Solecki]
- Groups with complete left invariant metrics [Becker]

The space of countable structures

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The following and their complements form a basis of clopen sets

- $\{\mathcal{G} \in \mathcal{X}_\mathcal{L} : a +_G b = c\}$ for $a, b, c \in \mathbb{N}$.
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Theorem: $\mathcal{X}_L$ is a Polish space.

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Theorem: If φ is a 1st-order sentence, $\{\mathcal{G} \in \mathcal{X}_L : \mathcal{G} \models \varphi\}$ is Borel.

Background on infinitary logic

Definition: $L_{\omega_1,\omega}$ is the infinitary first-order language,
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Theorem[Lopez-Escobar]: For $\mathbb{K} \subseteq \mathcal{X}_{\mathcal{L}}$ closed under isomorphisms,
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Lemma: [Scott 65] For every structure $\mathcal{A}$, there is an $L_{\omega_1,\omega}$ sentence φ
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The number of countable models of an $\mathcal{L}_{\omega_1,\omega}$ sentence
is either countable, or $2^{\aleph_0}$.

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For $\mathcal{A} \in \mathcal{X}_L$, $f \in S_\infty$, $f \cdot \mathcal{A}$ is the structure $\mathcal{B}$ such that

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- For any continuous action of S_∞ into any Polish space, the number of orbits in any Borel invariant set is either countable or continuum.

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The number of equivalence classes is either countable, $\aleph_1$ or $2^{\aleph_0}$.

Theorem ([M.] ZFC+PD + $\neg$CH)

Let $\equiv$ be an analytic equivalence relation on $\mathbb{R}$.
The following are equivalent:

- *There are $\aleph_1$ many $\equiv$-equivalence classes.*
- *The equivalence classes are linearly ordered by computability*

  on a cone, non-trivially.

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We called such a real a *presentation of the structure*.

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1. If $\mathbb{K}$ is an axiomatizable class of structures, then the set of presentation of structures in $\mathbb{K}$ is Borel.
2. The set of triples $(\mathcal{A}, \mathcal{B}, f) \in \mathbb{R}^3$, where $\mathcal{A}$ and $\mathcal{B}$ are presentations of structures in $\mathbb{K}$ and f is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$ is Borel.

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3. The set of pairs $(\mathcal{A}, \mathcal{B}) \in \mathbb{R}^3$, where $\mathcal{A}$ and $\mathcal{B}$ are presentations of isomorphic structures in $\mathbb{K}$ is analytic.

Analytic Sets

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There exists analytic equivalence relations with $\aleph_1$ equivalence classes.

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Def: Let $\mathcal{R}, \mathcal{S} \subseteq \mathbb{R}$.
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Obs: If $\mathcal{A}$ and $\mathcal{B}$ are structures then, $\mathcal{A}$ is computable in $\mathcal{B}$ iff and only if the set of presentations of $\mathcal{A}$ is computable in the set of presentations of $\mathcal{B}$.

Examples:

The following equivalence classes have $\aleph_1$-many equivalence classes:

1. isomorphism on well-orderings;
2. bi-embeddability on linear orderings;
3. bi-embeddability on torsion p-groups;
4. isomorphism on models of a counterexample to Vaught's conjecture when relativized;
5. $X \equiv Y \iff X$ and Y compute the same well-orderings.

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Thm [M., Greenberg 05]: For $\sim$ as in (3), computability is linear.

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