# Unsolvable problems, the Continuum Hypothesis, and the nature of infinity

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# $V$: The Universe of Sets

## The power set

*Suppose $X$ is a set. The powerset of $X$ is the set*

$$\mathcal{P}(X) = \{Y \mid Y \text{ is a subset of } X\}.$$

## Cumulative Hierarchy of Sets

*The universe $V$ of sets is generated by defining $V_\alpha$ by induction on the ordinal $\alpha$:*

1. $V_0 = \emptyset$,
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
3. *if $\alpha$ is a limit ordinal then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.*

- If $X$ is a set then $X \in V_\alpha$ for some ordinal $\alpha$.

- $V_0 = \emptyset$, $V_1 = \{\emptyset\}$, $V_2 = \{\emptyset, \{\emptyset\}\}$.
  - These are just the ordinals: 0, 1, and 2.
- $V_3$ has 4 elements.
  - This is not the ordinal 3 (in fact, it is not an ordinal).
- $V_4$ has 16 elements.
- $V_5$ has $65,536$ elements.
- $V_{1000}$ has a lot of elements.

*$V_\omega$ is infinite, it is the set of all (hereditarily) finite sets.*

*The conception of $V_\omega$ is* **mathematically identical** *to the conception of the structure $(\mathbb{N}, +, \cdot)$.*

# Beyond the basic axioms: large cardinal axioms

**The axioms**

- *The* ZFC *axioms of Set Theory specify the basic axioms for* $V$.
- *These axioms are naturally augmented by additional principles which assert the existence of "very large" infinite sets.*
  - *These additional principles are called* **large cardinal axioms**.

- *There is a proper class of measurable cardinals.*
- *There is a proper class of strong cardinals.*
- *There is a proper class of Woodin cardinals.*
- *There is a proper class of superstrong cardinals.*
- *There is a proper class of supercompact cardinals.*
- *There is a proper class of extendible cardinals.*
- *There is a proper class of huge cardinals.*
- *There is a proper class of* $\omega$*-huge cardinals.*

# Cardinality: measuring the size of sets

**Definition: when two sets have the same size**

*Two sets, $X$ and $Y$, have the same* **cardinality** *if there is a matching of the elements of $X$ with the elements of $Y$.*

*Formally: $|X| = |Y|$ if there is a bijection*

$$f : X \to Y$$

- $|\{0, 1, 2, \ldots, k, \ldots\}| = |\{1, 2, 3, \ldots, k, \ldots\}|$.
- $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$.
- $|\mathbb{R}| = |\mathbb{R}^2|$ (not obvious at all!).

Assuming the $\mathrm{ZFC}$ axioms:

**Theorem (Cantor)**

*For every set $X$ there is an ordinal $\alpha$ such that $|X| = |\alpha|$.*

# The Continuum Hypothesis: CH

**Theorem (Cantor)**

*The set $\mathbb{N}$ of all natural numbers and the set $\mathbb{R}$ of all real numbers do not have the same cardinality.*

**The Continuum Hypothesis**

*Suppose $A \subseteq \mathbb{R}$ is infinite. Then either:*

1. *$A$ and $\mathbb{N}$ have the same cardinality, or*
2. *$A$ and $\mathbb{R}$ have the same cardinality.*

- This is Cantor's Continuum Hypothesis.
- The Continuum Hypothesis holds for the simple infinite sets
  - and it holds for many not-so-simple infinite sets.

# Many tried to solve the problem of $\mathrm{CH}$ and failed.

*In 1940, Gödel showed that it is consistent with the axioms of Set Theory that the* Continuum Hypothesis *be true.*

- *One* **cannot refute** *the* Continuum Hypothesis*.*

*In 1963, on July 4th, Cohen announced in a lecture at Berkeley that it is consistent with the axioms of Set Theory that the* Continuum Hypothesis *be false.*

- *One* **cannot prove** *the* Continuum Hypothesis*.*

# Cohen's method

*If $M$ is a universe of Set Theory then $M$ contains "blueprints" for virtual universes $N$ which extend $M$. These blueprints can be constructed and analyzed from within $M$.*

- *If $M$ is countable then every blueprint constructed within $M$ can be realized as genuine extension of $M$.*

- Cohen proved that **every** universe $M$ contains a blueprint for an extension in which the Continuum Hypothesis is false.
- Cohen's method also shows that **every** universe $M$ contains a blueprint for an extension in which the Continuum Hypothesis is true.
- (Levy-Solovay) These extensions preserve large cardinal axioms if these axioms hold for proper class of cardinals.
  - So if large cardinal axioms can help
    - it can only be in some **indirect** way.

# The extent of Cohen's method: It is not just about CH

**A challenging time for the conception of $V$**

- *Cohen's method of forcing has been vastly developed.*
- *Many questions have been showed to be unsolvable.*

- A serious challenge to the very conception of Mathematical Infinity.
  - The Continuum Hypothesis is a statement about just $V_{\omega+2}$.

# We have a problem

**Question**

*Is there a resolution?*

- Perhaps one should begin by trying to understand $\mathrm{CH}$

**A natural conjecture**

*One can understand $\mathrm{CH}$ by looking at special cases.*

- But which special cases?
    - Does this even make sense?

# Seeking special cases to study for the problem of the Continuum Hypothesis

**Logical definability within a set $X$**

*Suppose that $X$ is a set. A subset $Y \subseteq X$ is logically definable in $X$ from parameters if there is a formal property $\varphi(x_0, \ldots, x_n)$ and elements $a_1, \ldots, a_n$ of $X$ such that*

- *$Y$ is the set of all $a \in X$ such that $\varphi[a, a_1, \ldots, a_n]$ holds in $X$.*

**The definable power set**

For each set $X$, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that $Y$ is logically definable in the structure $(X, \in)$ from parameters in $X$.

- $\mathcal{P}_{\mathrm{Def}}(X)$ is the collection of all "specifiable" subsets of $X$
  - versus $\mathcal{P}(X)$ which is the collection of **all** subsets of $X$.

# The projective sets

**Definition**

A set $A \subseteq \mathbb{R}$ is *projective* if it can be generated from the open subsets of $\mathbb{R}$ in finitely many steps of taking complements and images by continuous functions,

$$f : \mathbb{R} \to \mathbb{R}.$$

- The study of the projective sets was a major focus of the Polish School of Mathematics in the early 1900s.
- $\mathbb{R} \subset V_{\omega+1}$ and so $\mathcal{P}(\mathbb{R}) \subset \mathcal{P}(V_{\omega+1}) = V_{\omega+2}$.

*The collection of projective sets is exactly:*

$$\mathcal{P}(\mathbb{R}) \cap \mathcal{P}_{\mathrm{Def}}(V_{\omega+1})$$

# The Continuum Hypothesis and the Projective Sets

**The projective Continuum Hypothesis**

*Suppose $A \subseteq \mathbb{R}$ is an infinite projective set. Then either:*

1. *$A$ and $\mathbb{N}$ have the same cardinality, or*
2. *$A$ and $\mathbb{R}$ have the same cardinality.*

- There were many attempts in the early 1900s to prove the projective Continuum Hypothesis with success for the simplest instances.
- However, by 1925 it too began to look like a hopeless problem.

# It was a hopeless problem

*The actual constructions of Gödel and Cohen show that the projective Continuum Hypothesis is also formally unsolvable.*

- This explains why the problem of the projective Continuum Hypothesis was so difficult.
- But the intuition that the problem **has** an answer was correct.

**Theorem**

*Suppose there are infinitely many Woodin cardinals. Then the* **strong** *projective Continuum Hypothesis is true:*

- *Every uncountable projective set contains an uncountable closed set.*

- Maybe the problem of the Continuum Hypothesis also has an answer
  - and maybe the key is the definable powerset $\mathcal{P}_{\mathrm{Def}}(X)$.

# The effective cumulative hierarchy: $L$

## Cumulative Hierarchy of Sets

*The cumulative hierarchy is defined by induction on $\alpha$ as follows.*

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. *if $\alpha$ is a limit ordinal then* $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

- *$V$ is the class of all sets $X$ such that $X \in V_\alpha$ for some $\alpha$.*

## Gödel's constructible universe, $L$

*Define $L_\alpha$ by induction on $\alpha$ as follows.*

1. $L_0 = \emptyset$.
2. $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$.
3. *if $\alpha$ is a limit ordinal then* $L_\alpha = \cup\{L_\beta \mid \beta < \alpha\}$.

- *$L$ is the class of all sets $X$ such that $X \in L_\alpha$ for some $\alpha$.*

**The axiom: $V = L$**

*Suppose $X$ is a set. Then $X \in L$.*

**Theorem (Gödel:1940)**

*Assume $V = L$. Then the Continuum Hypothesis holds.*

- Suppose there is a Cohen-blueprint for $V = L$. Then:
  - the axiom $V = L$ must hold and the blueprint is trivial.

**Claim**

*Adopting the axiom $V = L$ completely negates the ramifications of Cohen's method.*

- Could this be the resolution?

# The axiom $V = L$ and large cardinals

**Theorem (Scott:1961)**

*Assume $V = L$. Then there are no measurable cardinals.*

- *Then there are no (genuine) large cardinals.*

- Assume $V = L$. **Then there are no Woodin cardinals.**

**Clearly:**

*The axiom $V = L$ is false.*

- We need to generalize $L$.
  - But how?

# Towards generalizing the projective sets

# More about the projective sets

## Definition

A set $A \subseteq \mathbb{R}^n$ is **projective** if it can be generated from the open subsets of $\mathbb{R}^n$ in finitely many steps of taking complements and images by continuous functions,

$$f : \mathbb{R}^n \to \mathbb{R}^n.$$

## Definition

Suppose that $A \subseteq \mathbb{R} \times \mathbb{R}$. A function $f$ **uniformizes** $A$ if for all $x \in \mathbb{R}$:

- if there exists $y \in \mathbb{R}$ such that $(x, y) \in A$ then $(x, f(x)) \in A$.

# Two questions of Luzin

## Two questions of Luzin

1. *Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is projective. Can $A$ be uniformized by a projective function?*
2. *Suppose $A \subseteq \mathbb{R}$ is projective. Is $A$ Lebesgue measurable and does $A$ have the property of Baire?*

*Luzin's questions are questions about* $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

*Luzin conjectured in 1925 that "we will never know the answer to the measure question for the projective sets".*

**Both questions are unsolvable on the basis of the $\mathrm{ZFC}$ axioms**

# Determinacy and the answers to Luzin's questions

Suppose $A \subseteq \mathbb{R}$. There is an associated infinite game involving two players.

- The players alternate choosing $\epsilon_i \in \{0, 1\}$.
- After infinitely many moves an infinite binary sequence $\langle \epsilon_i : i \in \mathbb{N} \rangle$ is defined.
- Player I wins this run of the game if

$$\Sigma_{i=1}^{\infty} \epsilon_i / 2^i \in A$$

otherwise Player II wins.

**Definition**

The set $A$ is **determined** if there is a winning strategy for one of the players in the game associated to $A$.

# The Axiom of Determinacy ($\mathrm{AD}$)

**Definition (Mycielski-Steinhaus)**

**Axiom of Determinacy** ($\mathrm{AD}$): Every set $A \subseteq \mathbb{R}$ is determined.

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**Lemma (Axiom of Choice)**

*There is a set $A \subset \mathbb{R}$ such that $A$ is not determined.*

**Corollary**

$\mathrm{AD}$ *is false.*

# Projective Determinacy (PD)

**Definition**

**Projective Determinacy** (PD): Every projective set $A \subseteq \mathbb{R}$ is determined.

**Theorem**

*Assume every projective set is determined.*

(1) (Mycielski-Steinhaus) *Every projective set has the property of Baire.*

(2) (Mycielski-Swierczkowski) *Every projective set is Lebesgue measurable.*

(3) (Moschovakis) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

**Key questions**

*Is PD even consistent and if consistent, is PD true?*

# Elementary embeddings

**Definition**

Suppose $X$ and $Y$ are transitive sets. A function $j : X \to Y$ is an **elementary embedding** if for all logical formulas $\varphi[x_0, \ldots, x_n]$ and all $a_0, \ldots, a_n \in X$,

$$(X, \in) \models \varphi[a_0, \ldots, a_n] \text{ if and only if } (Y, \in) \models \varphi[j(a_0), \ldots, j(a_n)]$$

- Isomorphisms are elementary embeddings but the only isomorphisms of $(X, \in)$ and $(Y, \in)$ are trivial.

**Lemma**

*Suppose that $j : X \to Y$ is an elementary embedding and that $X \models \mathrm{ZFC}$. Then the following are equivalent.*

(1) *$j$ is not the identity.*

(2) *There is an ordinal $\beta \in X$ such that $j(\beta) \neq \beta$.*

# Strong axioms of infinity: large cardinal axioms

**Basic template for large cardinal axioms**

*A cardinal $\kappa$ is a* **large cardinal** *if there exists an elementary embedding,*

$$j : V \to M$$

*such that $M$ is a transitive class and $\kappa$ is the least ordinal such that $j(\alpha) \neq \alpha$.*

- Requiring $M$ be *close* to $V$ yields a hierarchy of large cardinal axioms:
  - simplest case is where $\kappa$ is a *measurable cardinal*.
- $M = V$ contradicts the Axiom of Choice.

*The hierarchy of large cardinal axioms has emerged as the fundamental core of Set Theory.*

- *It is (empirically) a wellordered hierarchy and provides a calibration of the unsolvability of problems in Set Theory.*

# The validation of Projective Determinacy

**Theorem (Martin-Steel)**

*Assume there are infinitely many Woodin cardinals. Then every projective set is determined.*

**Theorem**

*The following are equivalent.*

(1) *Every projective set is determined.*

(2) *For each $n < \omega$ there is a countable (iterable) model $M$ such that $M \models \mathrm{ZFC} +$ "There exist $n$ Woodin cardinals".*

$\mathrm{PD}$ **is the missing (and true) axiom for** $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

- *Is there such an axiom for $V$ itself?*