MATROID PATH-WIDTH

DEF: The least k s.t. $E(M)=(x_1,x_2,\dots,x_n)$:
[GGW]

$$\forall i \quad \lambda_M(x_1,\dots,x_i) \leq k.$$

- Equivalent to linear branch-width, but
- not the same as the path version of matroid tree-width ?!

- Same as the trellis-width of a lin. code!

COMPUTING MATR. PW.
(at least, for $\mathbb{F}$-represented matroids)

<u>BASICS</u> – minor-monotone,
– bounding branch-width,

$\Rightarrow$ and hence, can use the alg. of [PH2005]:

① compute a 3-approx. branch-decomposition of $\mathbb{F}$-repres. M in $O(f(k)\cdot n^3)$,

② test the finite number of excluded minors for "$pw \leq k$" in $O(g(k)\cdot n)$.

* <u>Note</u>, $g(k)$ is not computable since we do not have a size bound on the minors!

COMPUTING MATR. PW. A DECOMPOSITION?

- The previous computes a number, not ordering.

THM: [Jeong-Kim-Oum, SODA 2016]
Can compute an optimal path-decomposit. of an $\mathbb{F}$-represented matroid in FPT.

+ GOOD: a constructive algorithm, uniform
− NOT SO GOOD: complicated, proof hard to follow

- Can also use the approach [PH-Oum 2008]?

SELF-REDUCTION FOR M.PW.

IDEA: Use repeated calls to a path-width (as number) oracle to find the ordering.

- e.g., self-reductions for 3-COL, HAM, ...

Here we need:

$\boxed{\text{ORACLE}_k}$ Input: $\mathbb{F}$-represented M, and a subspace Σ.

Question: Is there a path-decomp. of $M \cup \{\Sigma\}$ of width $\leq k$ starting with the "polyelement" Σ?

USING THE ORACLE

I ORACLE_k can construct a path d. $\leq k$.

① Start with the empty ordering $\delta = \emptyset$.

② Find exhaustively any $\underline{x} \in E(M) \setminus \delta$ s.t.: ORACLE_k says $\underline{\text{YES}}$ for $\underline{M \setminus \delta}$ and $\Sigma = \underline{\langle \delta \cup \{x\} \rangle}$.

③ Continue until $\delta = E(M)$, or stop with $\underline{\text{NO}}$ if no x exists.

PROOF: Straightforward, such $\underline{x}$ exists if pw $\leq k$, at every step by induction. Conv., $\underline{\lambda_M(\delta) \leq k}$ holds on success.

$\square$

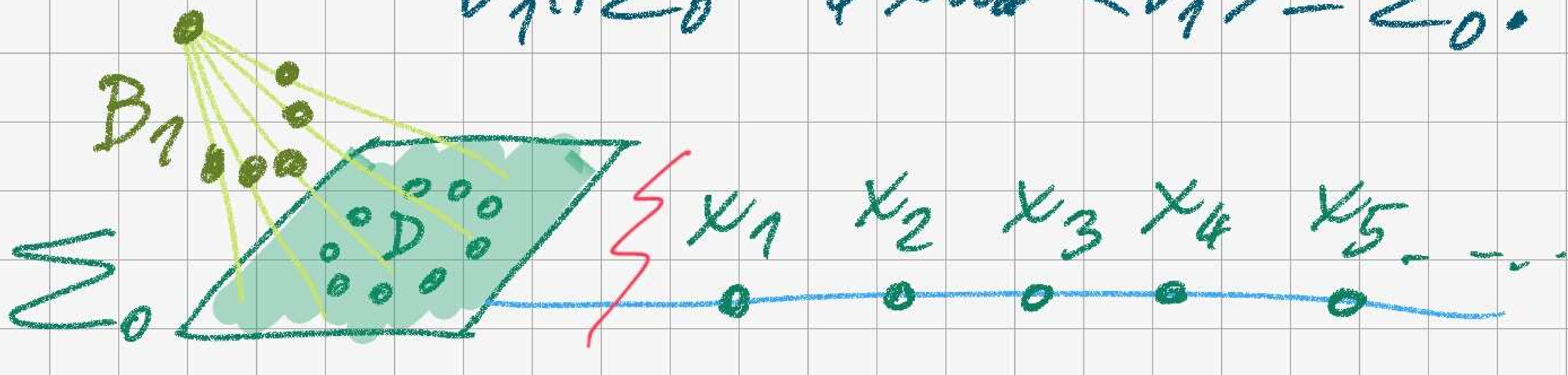
IMPLEMENTING IT

II ORACLE$_k$ can be implemented in FPT.

1. Input M and Σ over $\mathbb{F}$, and fix any $\underline{\underline{\Sigma_0}} \supseteq \Sigma \cap \langle E(M) \rangle$, $rk(\Sigma_0) = k$. (If nonexistent, answer $\underline{NO.}$)
2. get (sufficient) extension field $\mathbb{F}_0$ of $\mathbb{F}$. Fix any $D \subseteq \Sigma_0$, $D \cong \underline{U_{k,2k}}$.
3. Choose any basis B_0 of Σ_0 $\underline{\text{free}}$ wrt. M. Freely lift B_0 into $\underline{B_1 \text{ of rank } k+1.}$

IMPLEMENTING (contd.)

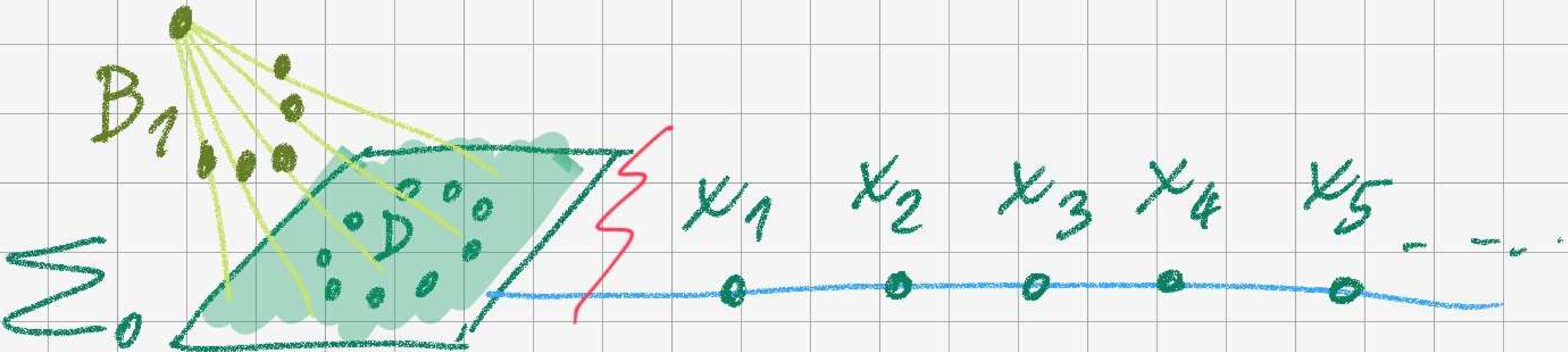
RECALL: fixed any $D \subseteq \Sigma_0$ and $D \cong U_{k,2k}$, freely chosen B_1, $|B_1| = rk(B_1) = k+1$ and $B_1 \cap \Sigma_0 = \emptyset$ but $\langle B_1 \rangle \supseteq \Sigma_0$.



④ ask the "numeric" pw. algorithm whether $\underline{pw(M \cup D \cup B_1) \leq k}$.

$\square$

PROVING THE ORACLE

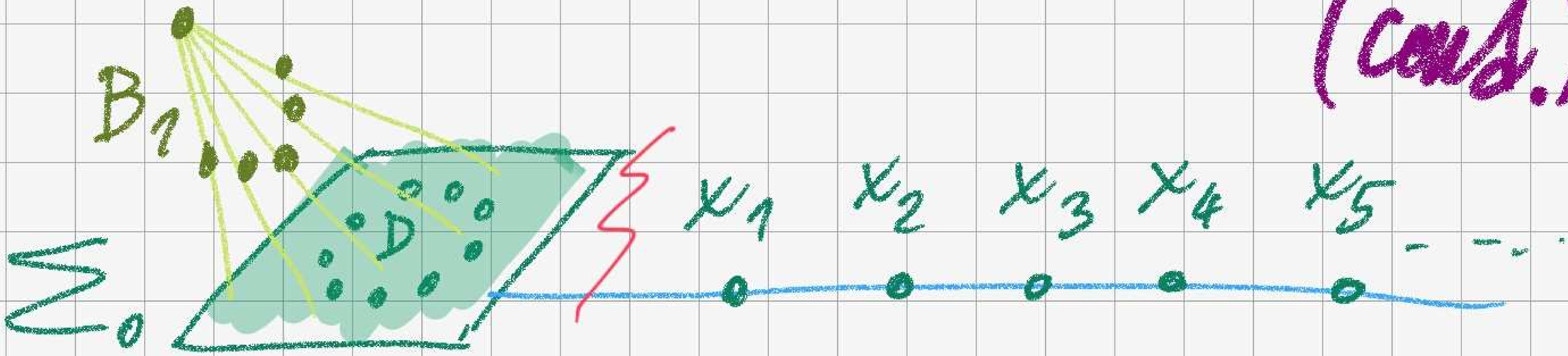


① $pw(\Sigma, x_1, \dots x_m) \leq k \implies$ ORACLE$_k$ says YES.

– filling in D and B_1 on the left, easily

② ORACLE$_k$ says YES $\implies$ . . . .

(cont.)



② ORACLE$_k$ says <u>YES</u> $\Longrightarrow$

a) $D \cong U_{k,2k}$ forces $\langle D \rangle = \Sigma_0$ as some <u>guts</u>,

b) $a, b \in E(M) \setminus \Sigma_0$ and $rk(D \cup \{a,b\}) = k+1$
$\Longrightarrow$ a, b to the <u>same side</u> of Σ_0 in order,

$\rightarrow$ all B_1 to the left of Σ_0.

c) $C \subseteq M \setminus \Sigma_0$ a circuit and $\langle C \rangle \supseteq \Sigma_0$
$\Longrightarrow$ <u>not</u> all C to the same side of Σ_0,

$\rightarrow$ all M to the right of Σ_0. $\boxtimes$

QUESTIONS

① Can we bound the size / number of excluded minors for path-width $\leq k$?

② What about $\mathbb{F}$-represented poly-matroids? (Jeong-Kim-Oum do this)

But, the path-width definition does not seem "right" for polymatroids.