Inequivalent representations of matroids with no $U_{3,6}$-minor

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With Jim Geelen and Geoff Whittle.

Equivalence of representations

Let M be a matroid. Let A and A' be matrices over a field, $\mathbb{F}$, that represent M. Columns are labelled with elements of $E(M)$.

A and A' are equivalent if one is obtained from the other by:

- adding a row to another,
- scaling rows/columns by numbers in $\mathbb{F} - \{0\}$,
- permuting rows,
- permuting columns and column labels,
- deleting/adding zero rows,
- applying an automorphism of $\mathbb{F}$ entrywise.

If M is GF(q)-representable, let $n_q(M)$ be the number of equivalence classes of matrices that represent M over GF(q).

Kahn's conjecture

Theorem (White – 1971)

$n_2(M) = 1$ for any GF(2)-representable matroid M.

Theorem (Brylawski and Lucas – 1976)

$n_3(M) = 1$ for any GF(3)-representable matroid M.

Kahn's conjecture

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Theorem (Brylawski and Lucas – 1976)

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Theorem (Kahn – 1988)

$n_4(M) = 1$ for any 3-connected GF(4)-representable matroid M.

Kahn's conjecture

Conjecture (Kahn – 1988)

Let q be any prime power. There exists an integer N_q such that

$$n_q(M) \leq N_q$$

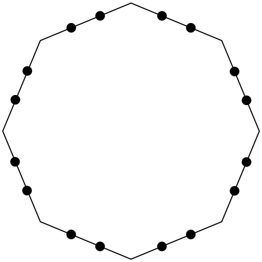
for any 3-connected GF(q)-representable matroid M.

Kahn's conjecture

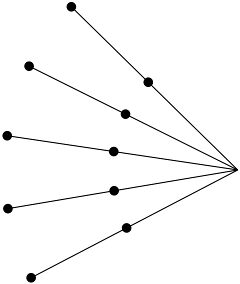
Theorem (Oxley, Vertigan, Whittle – 1996)

$n_5(M) \leq 6$ for any 3-connected GF(5)-representable matroid M.

Kahn's conjecture



free-swirl



free-spike

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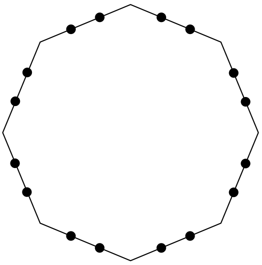

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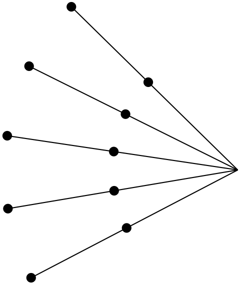
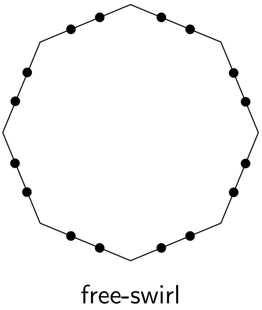


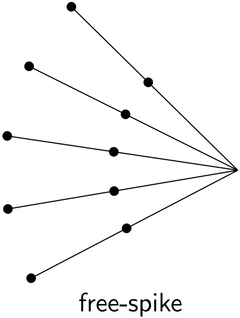
free-spike

Theorem (Oxley, Vertigan, Whittle – 1996)

Let q be a prime power with $q > 5$.
Assume $q - 1$ is composite. If M is the rank-r free-swirl, then M is GF(q)-representable and $n_q(M) \geq 2^r$.

Kahn's conjecture





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Assume $q-1$ is composite. If M is the rank-r free-swirl, then M is GF(q)-representable and $n_q(M) \geq 2^r$.
Assume $q-1$ is prime. If M is the rank-r free-spike, then M is GF(q)-representable and $n_q(M) \geq 2^{r-1}$.

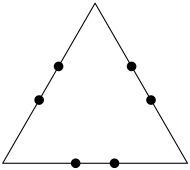
Kahn's conjecture

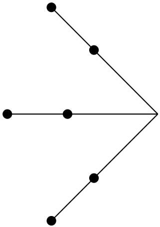
Conjecture (Geelen, Oxley, Vertigan, Whittle – 2002)

Let q be a prime power, and let $r \geq 3$ be an integer. There exists an integer $N_{q,r}$ such that if M is a 3-connected GF(q)-representable matroid with no minor isomorphic to the rank-r free-swirl or free-spike, then

$$n_q(M) \leq N_{q,r}.$$

Kahn's conjecture





Note that the rank-3 free-swirl and the rank-3 free-spike are both isomorphic to $U_{3,6}$.

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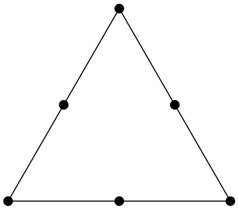
If such an M' exists with $\{e, e'\}$ independent, then e is free, otherwise e is fixed.



Fixed elements

Assume e is fixed in M, and both

$$\left[\begin{array}{c|c} A & \overset{e}{\mathbf{x}} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} A & \overset{e}{\mathbf{x}'} \end{array}\right]$$

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So in this case,

$$n_q(M) \leq n_q(M \backslash e).$$

If e is cofixed (fixed in M^*), then $n_q(M) \leq n_q(M/e)$.

Totally free matroids

Assume we want to bound $n_q(M)$ for a 3-connected $\mathrm{GF}(q)$-representable matroid M.

If M' is 3-connected, and is produced from M by a sequence of:

- deleting a fixed element, where the deletion is 3-connected up to series pairs,
- contracting a cofixed element, where the contraction is 3-connected up to parallel pairs,

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Definition

M' is totally free if $|E(M')| \geq 4$, M' is 3-connected, and $\mathrm{co}(M' \backslash e)$ is not 3-connected whenever e is fixed, and $\mathrm{si}(M'/e)$ is not 3-connected whenever e is cofixed.

Totally free matroids

Let $\mathcal{M}$ be a minor-closed class of matroids. Let q be a prime power.

If $\{M_1, \dots, M_n\}$ is the set of totally free GF(q)-representable matroids in $\mathcal{M}$, then

$$n_q(M) \leq \max\{n_q(M_1), \dots, n_q(M_n)\}$$

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Theorem (Geelen, Oxley, Vertigan, Whittle – 2002)

Let M be a totally free matroid with $|E(M)| \geq 5$. Either:

- $M \backslash e$ is totally free for some $e \in E(M)$,
- M/e is totally free for some $e \in E(M)$,
- $E(M)$ is a union of 2-element clonal classes, and $M \backslash e/e'$ is totally free for any clonal class $\{e, e'\}$.

Quasi-lines

A Δ-Y exchange replaces a triangle with a triad.

A segment-cosegment exchange replaces a k-element line with a k-element coline.

A quasi-line is produced by starting with $U_{2,k}$ ($k \geq 4$), and repeatedly applying segment-cosegment exchanges and the dual operation.

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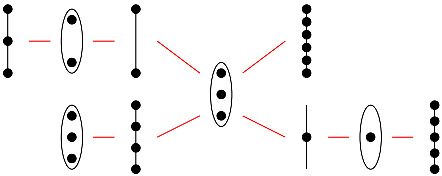
A quasi-line is produced by starting with $U_{2,k}$ ($k \geq 4$), and repeatedly applying segment-cosegment exchanges and the dual operation.

Theorem (Oxley, Semple, Vertigan – 2000)

Every quasi-line is uniquely described by a reduced del-con tree.



Quasi-lines

Theorem (Geelen, Mayhew, Whittle – 2004)

The following are equivalent:

- M is a totally free matroid with no $U_{3,6}$-minor,
- M is a quasi-line.

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- M has no triangles and no triads.

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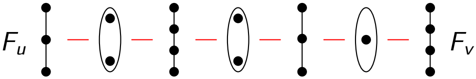
- M has no triangles and no triads.
- Up to duality, there is an element e such that $M\backslash e$ is totally free, and hence a quasi-line.
- We prove that M/e is also totally free, and hence a quasi-line.

Proof

- Consider a longest path in the reduced del-con tree corresponding to M/e.



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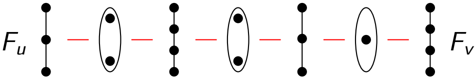

- The ends of the path are lines, since M and M/e have no triads. Let these lines be F_u and F_v.

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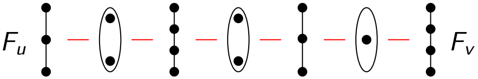

- The ends of the path are lines, since M and M/e have no triads. Let these lines be F_u and F_v.
- Since M has no triangles, $F_u \cup e$ and $F_v \cup e$ are rank-3 cyclic flats of M.

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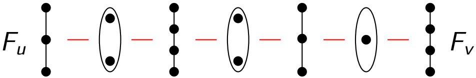

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- If $r(F_u \cup F_v \cup \{e\}) = 5$, then e is fixed, a contradiction, as $M \backslash e$ is 3-connected.

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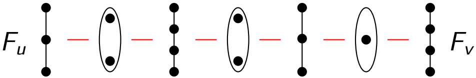

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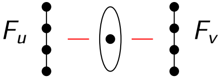
- If $r(F_u \cup F_v \cup \{e\}) = 5$, then e is fixed, a contradiction, as $M \backslash e$ is 3-connected.
- Thus $r(F_u \cup F_v \cup \{e\}) = 4$, and M/e is represented by this tree.



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- We have proved $r(M) = 4$. By duality, $r(M^*) = 4$, and $|E(M)| = 8$.
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- The rest is straightforward case-analysis.

Representations of a quasi-line are in correspondence with the rank-2 uniform matroid from which it is constructed.

Theorem (Geelen, Mayhew, Whittle – 2004)

Let M be a 3-connected GF(q)-representable matroid with no $U_{3,6}$-minor. Then

$$n_q(M) \leq n_q(U_{2,q+1}) = (q-2)!$$

The general conjecture

Conjecture (Geelen, Oxley, Vertigan, Whittle – 2002)

Let q be a prime power, and let $r \geq 3$ be an integer. There exists an integer $N_{q,r}$ such that if M is a 3-connected GF(q)-representable matroid with no minor isomorphic to the rank-r free-swirl or free-spike, then

$$n_q(M) \leq N_{q,r}.$$

The general conjecture

Theorem (Geelen, Whittle – 2013)

If p is a prime, then there is an integer N_p such that

$$n_p(M) \leq N_p$$

for every 4-connected GF(p)-representable matroid M.

The general conjecture

This conjecture follows as a corollary.

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Let q be a prime power, and let $r \geq 3$ be an integer. There exists an integer $N_{q,r}$ such that if M is a 3-connected GF(q)-representable matroid with no minor isomorphic to the rank-r free-swirl or free-spike, then

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