TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI
VICTORIA UNIVERSITY OF WELLINGTON

Run offs and run outs: extending the life table to other disciplines

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Content of the presentation

- Three disciplines into which progression of life tables extends
  - distinct mechanisms for generating ‘time series-like’ development for a single ‘cohort’
  - mechanisms for generating new cohorts
  - commonalities of statistical methodologies
  - miscellaneous comments
- Comments on Estate’s new book
  - Statistical methods with applications to demography and insurance

Data progression

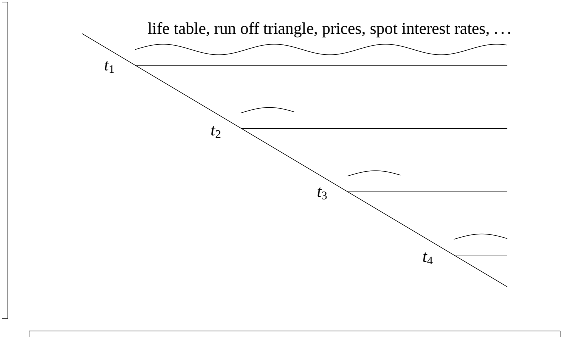

Figure: 1. Progression of time series like data

We consider mathematical/statistical modelling

- of data in the form of ‘time series’;
- progressing over time

The data assumes the form of

1. point processes
   - life tables: numbers of decrements
     - human deaths in demography
     - number of claims settled in insurance
     - machine breakdowns in reliability theory
     - numbers of bond defaults in finance
2. marked point processes
   - run off triangles in insurance
   - costs of, recovery rates for, bond defaults
   - cost of machine repairs in reliability theory

3. term structure of markets

Data assumes the form of ‘time series’, but is not really a time series.

Data merely summarises market prices/rates at a given time.

- term structure of market prices at a given time
    - spot/forward exchange rates
    - spot/forward prices: stocks/shares/indices
- term structure of interest rates
    - in the nature of a bank account offering ‘term’ interest rates
    - spot interest rates of varying terms
        - forward interest rates inferred from the spot rates
    - eg: libor zero (coupon bonds yield) curve
        - from Eurodollar futures market
        - manipulation thereof leading to fines of Barclays etc
        - roughly the subject matter of actuarial course CT8

Life Table

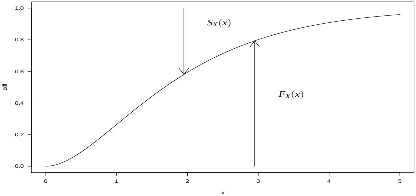

Figure: 2. CDF $F_X(x)$ and Survival Function $S_X(x)$

The random variable X is the waiting time until an event

The (Cumulative) Distribution Function (cdf) is $F_X(x) = \text{Prob}(X \leq x)$

The Probability Density Function (pdf, density) of X is $f_X(x) = F'_X(x)$

The survival function is $1 - F_X(x) = S_X(x)$.

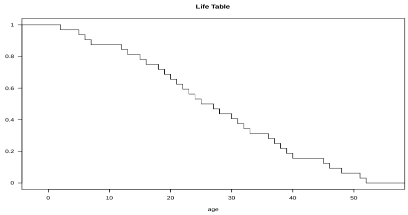

Figure: 3. The life table is the empirical survival function

The life table is the empirical survival function $\hat{S}_X(x) = \ell_X(x) = \ell_x$

The force of mortality $\mu_X(x)$, or the hazard rate $\lambda_X(x)$, is defined as

$$\lambda_X(x) = -\frac{S_X'(x)}{S_X(x)} = \frac{f_X(x)}{1-F_X(x)}$$

and $\lambda(t)dt$ has the interpretation of the probability of an event occurring in the next instant $(t, t+dt)$, *conditional* upon its not having occurred so far.

The hazard rate λ is variously called

- the force of mortality;
    - in demography
- the force of decrement
    - in actuarial science/demography
- the intensity (function)
    - in economics and finance generally; and particularly
    - in jump diffusion models in finance

and is analogous both to

- the continuously compounded rate of interest;
    - called the force of interest δ by actuaries
    - trade-off between δ and μ useful
        - eg, in dealing with impairment in life contingencies.

and further analogous to

- the instantaneous forward interest rate in term structure models of interest rates

For $S_X(x)$ and $\lambda_X(x)$, the central relationship is

$$S_X(x) = \exp\left(-\int_0^x \lambda_X(y)\,dy\right)$$

Let fv = future value (of a bank account) or fund value; also pv = present value

$$fv(t) = \exp\left(\int_0^t \delta(s)\,ds\right) \qquad \text{and} \qquad pv = \frac{1}{fv}$$

For the term structure of interest rates, f(t) = instantaneous forward rate (p.a.)

$$fv(t) = \left(1 + \frac{i_t}{2}\right)^{2t} = \exp\left(\int_0^t f(s)\,ds\right)$$

and where i_t is the (term) interest rate over $(0, t)$, semi-annually compounded (because bonds usually have a semi-annual coupon).
The graph of i_t against t is the 'term structure of interest rates'.
This is the 'yield curve'.

Transforming the x axis

- Dilate or contract time for a Poisson process of events
- ‘operational’ time
- subordination

Generation of one cohort from a previous cohort:

- happenchance is possible; but
- regeneration is often part of the modelling exercise
    - fertility
- change of state, from one life table to another
    - eg, getting sick

Statistical investigation of life tables

- horizontal
  - longitudinal study
- vertical
  - survey
  - progression of stable populations
- diagonal
  - continuing survey of population of constant age
- of special interest: what is μ?
  - variation of μ over life
    - how does μ behave for high ages?
  - variation of μ with illness, socio-economic status etc.
  - curve of deaths $\ell_x \mu_x$

Stochastic processes

An important broadening of perspectives

From the y axis to the x axis

- From counting decrements; to
- a stochastic process of events over time
- a ‘Point Process’

In this way we access the rich theory of stochastic processes

Some analogy with change from Riemann integral to Lebesgue integral

Run-off triangles

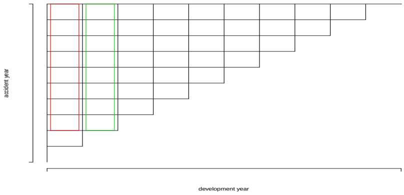

Figure: 4. The run-off triangle in insurance

Vertical axis is accident year, horizontal axis is development year.
But what is in the cells of the run-off triangle?

- claim counts
    - most amenable to mathematical modelling, particularly generalised linear models (GLM) type regression models
- average claim sizes
- total claims incurred
    - most used in practice

Insurance run-off is more complicated than the life table.
Consider the expected loss = EL:

$$\text{EL} = \text{exposure} \times \text{ppnl loss} \times \text{frequency} = \text{severity} \times \text{frequency}$$

Frequency will give one a point process; the severity will give a marked point process, for each cohort of claims.

- event = claims payment
- mark = payment made
- ambiguities remain: eg,
  - time of incurrence of claim often unclear
  - multiple payments on long tailed outstanding claims
  - inflation

The new cohort is not directly generated by the previous cohort.
Each row has its own exposure (to risk); and the hazard rate is not a useful concept, because we do not know the total claims to be paid.

The chain ladder method or variants is the main predictive tool.
The prediction factor is

$$\frac{\text{sum of elements in green}}{\text{sum of elements in red}}$$

to predict the next unknown cell.

Problem: no unique model driving the numerical algorithm

Work on finding models, for which the chain ladder gives maximum likelihood estimates, has been done by Verrall and Taylor, i.a.

These models can be considered as GLM type models.

Term structure of interest rates

Example: NZ mortgage rates (data from NZ Reserve Bank)

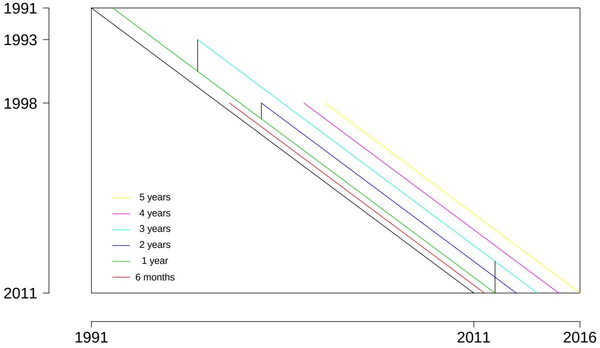

Figure: 5. Term structure of interest rates: mortgage rates in NZ

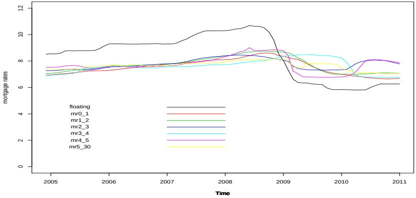

Figure: 6. Mortgage rates in NZ: view along the diagonals

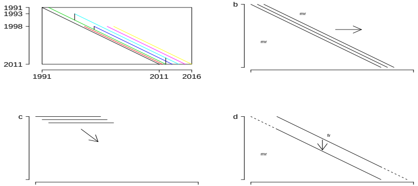

Figure: 7. Mortgage rates in NZ: how to model?

Characteristics of yield curve data

- Data has dimension of time, but is not really time series
- Ambiguities, i.a.:
  - coupon or zero coupon bonds? how risky?
  - theory usually works with zero coupon, risk-free bonds
    - libor no longer risk free
- generation of new ‘cohort’
  - updating expectations of future market rates/conditions
    - updating usually frequent (often daily)
  - test expectations theory
    - regression type models
    - residuals often autocorrelated
  - possible discrepancy between probability measures used
    - forward looking, pricing, using risk neutral measure Q
    - calibration, hypothesis testing, backwards: use real P measure
    - relation between P and Q depends on price of risk
      How to measure risk?

- often irregular ‘gappy’ data horizontally
    - preliminary statistical smoothing/filling in often needed
        - called ‘backfilling’ or ‘bootstrapping’, confusingly
        - need ‘smooth’ yield curve in order to define instantaneous forward rate
- data in form of prices
    - plot of spot/forward FX or stock prices vs ‘time’
    - reducible in theory to yield curve form of data
        - interest rates - yield - return

$$\text{return} = \frac{\Delta P + \text{divs}}{P} \to \frac{\Delta P}{P} \approx \Delta(\ln P)$$

        - this is rather artificial
    - but this ties in with financial theory, now based strongly on volatility
        - volatility = standard deviation of return
        - standard basic (Merton) model for return:

$$\frac{dP}{P} = \mu\, dt + \sigma\, dW_t$$

Comparison of three areas of application

Some of the main points are summarised thus:

- life tables and term structure: hazard rate vital
- run-off and term structure: regression type models
- term structure: not really time series
- Generation of new cohorts
    - life tables: often intrinsic part of model
    - run-off: 'random' new cohort
    - term structure: updating market expectations

Estate's book: Statistical methods with applications to demography and insurance

Advantages

- Well designed and presented
- rigorous
  - but emphasis is not on rigour
- presentation of mathematics exemplary
  - firm distinction between:
    - data generating process; and
    - empirical realisations
  - relatively advanced mathematics used
    - intuition emphasised
    - talking around the mathematics
  - always grounded in reality, eg:
    - duration of reigns of emperors

- a sensible mix between the familiar and the unfamiliar
  - extending the reader's knowledge of familiar topics
  - enhancing the reader's access to more powerful mathematics, i.a.
    - stochastic processes
    - Brownian motion/bridge
    - subordinated processes
- challenging; but enormously rewarding

Disdvantages

- none at all
- well, a few minor ones
    - some discussion of recent developments in the insurance and actuarial markets
    - greater cognisance taken of varying interest rates
        - stochastic
        - deterministic

On second thoughts, there is a major disadvantage to the book:

The pronunciation of the author’s name

Thank You.