Algorithmic Randomness

Denis R. Hirschfeldt — University of Chicago

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"How dare we speak of the laws of chance?
Is not chance the antithesis of all law?"

— Joseph Bertrand, Calcul des Probabilités, 1889

Part 1: Three Approaches to Defining Randomness

Computability Theory

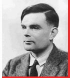
A First Look at Randomness

The Statistician's Approach: Martin-Löf Randomness

The Coder's Approach: Kolmogorov complexity

The Gambler's Approach: Martingales

Part 1: Three Approaches to Defining Randomness



Computability Theory

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Example: The set of primes is computable.

On input $n > 0$, run through all $1 < m \leqslant \sqrt{n}$.
  For each m, check whether m divides n.
    If some m does, return 0.
    If no m does, return 1.

Uniformly Computable Functions

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Then g is computable, so $f_e = g$ for some e.

But then $f_e(e) = g(e) = f_e(e) + 1$, a contradiction. □

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If $f(x)\downarrow$ for all $x \in \mathcal{X}$, then f is total.

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f is a partial computable function if there is an algorithm that on input x outputs $f(x)$ if $f(x)\downarrow$ and does not halt if $f(x)\uparrow$.

Universal Partial Computable Functions

We can list all partial computable functions $\mathbb{N} \to \mathbb{N}$ as $\Phi_0, \Phi_1, \ldots$ so that there is a single algorithm that on input (e, n) outputs $\Phi_e(n)$ if $\Phi_e(n)\downarrow$ and does not halt if $\Phi_e(n)\uparrow$.

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U is a universal partial computable function.

The definition of U depends on the choice of listing, but U's basic properties do not.

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The Halting Problem $\emptyset'$ is $\{(e, n) : \Phi_e(n)\downarrow\}$.

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Then $f_0, f_1, \ldots$ is a uniformly computable listing of all total computable functions, a contradiction. □

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There is a uniformly c.e. listing of all c.e. sets.

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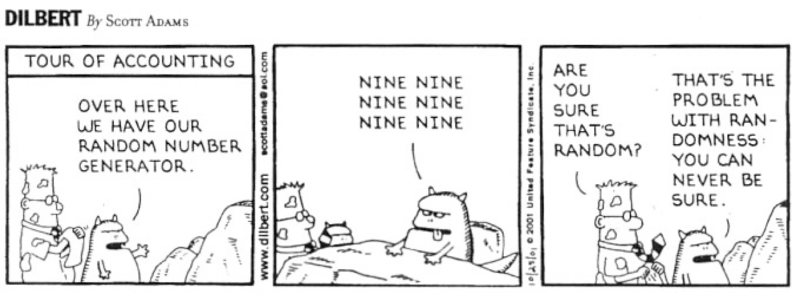

Intuitive Randomness

Which of the following binary sequences seem random?

A 000000000000000000000000000000000000000000000000000000000000

B 001101001101001101001101001101001101001101001101001101001101

C 010001101100000101001110010111011100000001001000110100010101

D 001001101101100010001111010100111011001001100000001011010100

E 010101110110111101110010011010110111001101101000011011110111

F 011101111100110110011010010000111111001101100000011011010101

G 000001100010111000100000000101000010110101000000100000000100

H 010100110111101101110101010000010111100000010101110101010001

Intuitive Randomness

Non-randomness: increasingly complex patterns.

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Randomness: bits coming from atmospheric patterns.

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Partial Randomness: mixing random and nonrandom sequences.

A 000000000000000000000000000000000000000000000000000000000000

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C 010001101100000101001110010111011100000001001000110100010101

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E 010101110110111101110010011010110111001101101000011011110111

F 011101111100110110011010010000111111001101100000011011010101

G 000001100010111000100000000101000010110101000000100000000100

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Intuitive Randomness

Randomness relative to other measures: biased coins.

A 000000000000000000000000000000000000000000000000000000

B 001101001101001101001101001101001101001101001101001101

C 010001101100000101001110010111011100000001001000110100010101

D 0010011011011000100011110101001110110010011000000001011010100

E 0101011101101111011100100110101101110011011010000110111101111

F 01110111110011011001101001000011111100110110000000110110101010

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We need a way to distinguish rare patterns from common patterns.

Three Approaches to Randomness at an Intuitive Level

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We begin by looking at an early attempt to define random sequences, by von Mises.

This attempt predated computability theory.

We will see how each of the three approaches above can be seen as an elaboration on von Mises' flawed attempt.

Von Mises' Approach

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Von Mises' basic idea: A gambler should not be able to make any money on a random sequence.

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Von Mises proposed that this observation could be turned around to characterize randomness.

Von Mises Randomness

A place selection rule is an increasing function $f : \mathbb{N} \to \mathbb{N}$, telling us which bits of a sequence to look at.

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Let $\mathcal{C}$ be a collection of place selection rules.

α is $\mathcal{C}$-von Mises random if $\lim_n R_n^f(\alpha) = \frac{1}{2}$ for all $f \in \mathcal{C}$.

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Church suggested taking $\mathcal{C}$ to be the *computable* place selection rules.

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Thm (Ville). Let $\mathcal{C}$ be any countable collection of place selection rules. There is a $\mathcal{C}$-von Mises random sequence α s.t. for all n,

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But how do we know this added requirement would be enough?

Three Approaches to Improving on von Mises' Idea

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Common solution: Use computability theory to define robust classes of tests, description systems, and betting strategies.

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Cantor Space and Effectively Open Sets

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For $B \subseteq 2^{<\omega}$, let $[B] = \bigcup_{\sigma \in B} [\sigma]$. Every open set in 2^ω is of this form.

We call B a set of generators for $[B]$.

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For $B \subseteq 2^{<\omega}$, let $[B] = \bigcup_{\sigma \in B} [\sigma]$. Every open set in 2^ω is of this form.

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No computable sequence can be 1-random.

Universal Martin-Löf Tests

We can list all ML-tests as

$$\begin{array}{l} \mathcal{C}^0_0, \mathcal{C}^0_1, \mathcal{C}^0_2 \cdots \\ \mathcal{C}^1_0, \mathcal{C}^1_1, \mathcal{C}^1_2 \cdots \\ \mathcal{C}^2_0, \mathcal{C}^2_1, \mathcal{C}^2_2 \cdots \\ \vdots \end{array}$$

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Let $\mathcal{U}_n = \bigcup_i \mathcal{C}_{i+n+1}^i$.

Then $\mathcal{U}_0, \mathcal{U}_1, \ldots$ is a ML-test, and α is 1-random iff it passes this single test.

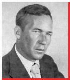
We call $\mathcal{U}_0, \mathcal{U}_1, \ldots$ a universal Martin-Löf test.

Part 1: Three Approaches to Defining Randomness

Computability Theory

A First Look at Randomness

The Statistician's Approach: Martin-Löf Randomness



The Coder's Approach: Kolmogorov complexity

The Gambler's Approach: Martingales

Kolmogorov Complexity

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The idea is to think of partial computable functions as systems of descriptions.

Plain Kolmogorov Complexity

Let $f : 2^{<\omega} \to 2^{<\omega}$ be partial computable.

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$$C_f(\sigma) = \min\{|\tau| : f(\tau) = \sigma\}.$$

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The plain Kolmogorov complexity of σ is $C(\sigma) = C_f(\sigma)$.

For every partial computable g, we have $C(\sigma) \leqslant C_g(\sigma) + O(1)$.

In particular, if f and g are both universal partial computable functions, then $C_f(\sigma) = C_g(\sigma) \pm O(1)$, so the definition of C does not depend on the choice of f, up to an additive constant.

Random Strings

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We might expect every initial segment of a random sequence to be random, and indeed want to characterize randomness of α by

$$C(\alpha \restriction n) \geqslant n - O(1).$$

However:

Thm (Martin-Löf). There is no $\alpha \in 2^\omega$ s.t. $C(\alpha \restriction n) \geqslant n - O(1)$.

A Criticism of Plain Kolmogorov Complexity

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Put another way, to describe binary strings, we use binary strings *plus* termination information.

A partial function $f : 2^{<\omega} \to 2^{<\omega}$ is prefix-free if its domain is an antichain, that is, if $f(\sigma)\downarrow$ and $\sigma \prec \tau$ or $\tau \prec \sigma$, then $f(\tau)\uparrow$.

Using only prefix-free partial computable functions as description systems gets around the above criticism.

Prefix-free Kolmogorov Complexity

List the prefix-free partial computable functions $f_0, f_1, \ldots$ and let

$$U(0^e 1\sigma) = f_e(\sigma).$$

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K is not computable, but it is computably approximable from above, i.e., there is a computable $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{N}$ s.t. $g(\sigma, n) \geqslant g(\sigma, n+1)$ and $\lim_n g(\sigma, n) = K(\sigma)$.

Prefix-Free Sets of Generators

Every open set $\mathcal{C}$ can be written as $[B]$ for some prefix-free $B \subset 2^{<\omega}$.

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In particular, for each σ, let σ^* be a minimal length string s.t. $U(\sigma^*) = \sigma$.

Then $\sum_{\sigma} 2^{-K(\sigma)} = \sum_{\sigma} 2^{-|\sigma^*|} \leqslant \sum_{\tau \in \mathrm{dom}\, U} 2^{-|\tau|} \leqslant 1$.

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In fact, $\mathcal{C}_0, \mathcal{C}_1, \ldots$ is a universal ML-test.

Bespoke Description Systems

For the other direction of Schnorr's Theorem, we need the following result.

KC Thm. Let $\langle n_i, \sigma_i \rangle_{i \in \mathbb{N}}$ be a computable sequence s.t. $\sum_i 2^{-n_i} \leqslant 1$.

There is a prefix-free partial computable f s.t.

$$\forall i \, \exists \tau_i \, (|\tau_i| = n_i \, \wedge \, f(\tau_i) = \sigma_i).$$

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The proof is a little messy, but f is easy to specify:

For each i, let τ_i be the leftmost string of length n_i incomparable with every τ_j for $j < i$, and let $f(\tau_i) = \sigma_i$.

1-randomness via Kolmogorov Complexity Revisited

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Proof of the $\Leftarrow$ direction. Let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be a universal ML-test.

There are uniformly c.e. sets $\{\sigma^0_0, \sigma^0_1, \ldots\}$, $\{\sigma^1_1, \sigma^1_1, \ldots\}$, $\ldots$ s.t. $\{\sigma^i_0, \sigma^i_1, \ldots\}$ is a prefix-free set of generators for $\mathcal{U}_i$.

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Consider the set of requests $\langle |\sigma_j^{2i+1}| - i, \sigma_j^{2i+1} \rangle$ for all i and j.

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$$\sum_{i,j} 2^{-(|\sigma_j^{2i+1}|-i)} = \sum_i 2^i \sum_j 2^{-|\sigma_j^{2i+1}|} = \sum_i 2^i \mu(\mathcal{U}_{2i+1}) \leqslant$$
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Proof of the $\Leftarrow$ direction. Let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be a universal ML-test.

There are uniformly c.e. sets $\{\sigma_0^0, \sigma_1^0, \ldots\}$, $\{\sigma_1^1, \sigma_1^1, \ldots\}$, $\ldots$ s.t. $\{\sigma_0^i, \sigma_1^i, \ldots\}$ is a prefix-free set of generators for $\mathcal{U}_i$.

Consider the set of requests $\langle |\sigma_j^{2i+1}| - i, \sigma_j^{2i+1} \rangle$ for all i and j.

$$\textstyle\sum_{i,j} 2^{-(|\sigma_j^{2i+1}|-i)} = \sum_i 2^i \sum_j 2^{-|\sigma_j^{2i+1}|} = \sum_i 2^i \mu(\mathcal{U}_{2i+1}) \leqslant$$
$$\textstyle\sum_i 2^i 2^{-(2i+1)} = \sum_i 2^{-(i+1)} = 1.$$

So by the KC Thm, $K(\sigma_j^{2i+1}) \leqslant |\sigma_j^{2i+1}| - i + O(1)$ for all i and j.

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So by the KC Thm, $K(\sigma_j^{2i+1}) \leqslant |\sigma_j^{2i+1}| - i + O(1)$ for all i and j.

If α is not 1-random, then $\alpha \in \mathcal{U}_{2i+1}$ for all i, so $\forall i \, \exists j, n \, (\sigma_j^{2i+1} = \alpha \upharpoonright n)$.

Thus $\forall i \, \exists n \, (K(\alpha \upharpoonright n) \leqslant n - i)$. $\square$

Part 1: Three Approaches to Defining Randomness

Computability Theory

A First Look at Randomness

The Statistician's Approach: Martin-Löf Randomness

The Coder's Approach: Kolmogorov complexity

The Gambler's Approach: Martingales

Martingales and Supermartingales

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We can replace d by a closely related martingale $\hat{d}$ s.t. $S_{\hat{d}} = S_d$ and $\liminf_n d(\alpha \upharpoonright n) = \infty$ for all $\alpha \in S_{\hat{d}}$.

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A supermartingale is a function $d : 2^{<\omega} \to \mathbb{R}^{\geqslant 0}$ s.t.

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An Example

Recall that $R_n(\alpha) = \frac{|\{m < n \,:\, \alpha(m) = 1\}|}{n}$.

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$$d(\alpha \upharpoonright n) = \left(\tfrac{1}{2}\right)^{n - nR_n(\alpha)} \left(\tfrac{3}{2}\right)^{nR_n(\alpha)} \geqslant O\left(\left(\tfrac{1}{2}\right)^{\frac{n}{3}} \left(\tfrac{3}{2}\right)^{\frac{2n}{3}}\right) = O\left(\left(\tfrac{1}{2}\tfrac{9}{4}\right)^{\frac{n}{3}}\right) = O\left(\left(\tfrac{9}{8}\right)^{\frac{n}{3}}\right).$$

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So $\lim_n d(\alpha \restriction n) = \infty$, and hence $\alpha \in S_d$.

Left-c.e. Reals and Functions

A real number x is left-c.e. if it can be computably approximated from below.

That is, there is a computable $f : \mathbb{N} \to \mathbb{Q}$ s.t. $f(n) \leqslant f(n+1)$ and $\lim_n f(n) = x$.

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A function $d : 2^{<\omega} \to \mathbb{R}$ is left-c.e. if there is a computable $f : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}$ s.t. $f(\sigma, n) \leqslant f(\sigma, n+1)$ and $\lim_n f(\sigma, n) = d(\sigma)$.

In other words, the values $d(\sigma)$ are uniformly left-c.e.

1-randomness via Martingales

Thm (Schnorr). The following are equivalent.

$\alpha \in 2^\omega$ is 1-random.

No left-c.e. martingale succeeds on α.

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Easier to see for supermartingales, because we can nicely list all left-c.e. supermartingales $d_0, d_1, \ldots$ and let

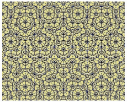
$$u(\sigma) = \sum_n 2^{-n} \frac{d_n(\sigma)}{d_n(\lambda)}.$$

Part 2: Examples, Properties, and Variations

Weakening 1-randomness

A Little More Computability Theory

Strengthening 1-randomness



Highly Nonrandom Sequences

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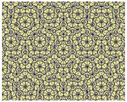

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Schnorr thought that computable randomness is not effective enough.

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Comparing Randomness Notions

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Nonmonotonic Randomness

A nonmonotonic betting strategy is one that, given α:

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This concept can be formalized using a nonmonotonic version of martingales.

α is nonmonotonically random if no computable nonmonotonic betting strategy makes arbitrarily much money betting on α.

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Open Question. Is every nonmonotonically random sequence 1-random?

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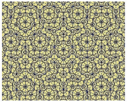

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Here is an algorithm showing that $\emptyset' \leqslant_{\mathrm{T}}$ TOT.

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Then $(e, n) \in \emptyset'$ iff $i \in$ TOT,
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If $f \leqslant_{\mathbf{T}} g$ and $g \leqslant_{\mathbf{T}} f$, then we say that f and g are Turing equivalent and write $f \equiv_{\mathbf{T}} g$.

Relativization

We can relativize other computability theoretic concepts.

For instance, A is B-c.e. if there is an algorithm for enumerating A using information from B.

Similarly, we can list all the A-partial computable functions $\Phi_0^A, \Phi_1^A, \ldots$ and define the Halting Problem relative to A as $A' = \{(e, n) : \Phi_e^A(n)\downarrow\}$.

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We can also relativize the notions of ML-test, prefix-free Kolmogorov complexity, and left-c.e. martingale and use these to define a notion of relativized 1-randomness.

For example: An A-Martin-Löf Test is a sequence of uniformly Σ_1^A classes $\mathcal{C}_0, \mathcal{C}_1, \ldots$ s.t. $\mu(\mathcal{C}_n) \leqslant 2^{-n}$.

α is A-1-random if $\alpha \notin \bigcap_n \mathcal{C}_n$ for every such test.

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A Σ_1^0 set is one of the form $\{n : \exists x\, R(n, x)\}$ with R a computable predicate.

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Let $\emptyset^{(n)} = (\emptyset^{(n-1)})'$.

$\emptyset^{(n)}$ is Σ^0_n, and every Σ^0_n set is $\emptyset^{(n)}$-computable.

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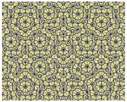

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Indeed, Ω can be seen as a highly compressed version of $\emptyset'$.

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Ω is the halting probability of U.

Ω is a left-c.e. real, and $\Omega \equiv_{\mathrm{T}} \emptyset'$.

Indeed, Ω can be seen as a highly compressed version of $\emptyset'$.

Ω is 1-random.

The Kučera-Gács Theorem

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In a sense, they are “fake 1-random sequences”.

Intuitively, we should not be able to extract information from random sequences, so they should be computationally weak.

Indeed, computing a given noncomputable set is a rare property.

Thm (de Leeuw, Moore, Shannon, and Shapiro; Sacks). If A is not computable then $\mu(\{B : A \leqslant_{\mathbf{T}} B\}) = 0$.

Strengthening 1-randomness

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$\alpha \in 2^\omega$ passes this test if $\alpha \notin \bigcap_n \mathcal{C}_n$.

α is weakly 2-random if it passes every generalized test.

n-randomness and Kolmogorov complexity

It is possible to characterize 2-randomness using Kolmogorov complexity.

Thm (Nies, Stephan, and Terwijn; Miller). α is 2-random iff $\exists^\infty n\,(C(\alpha \upharpoonright n) \geqslant n - O(1))$.

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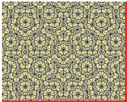
Open Problem. Are there characterizations along these lines for higher levels of randomness?

Part 2: Examples, Properties, and Variations

Weakening 1-randomness

A Little More Computability Theory

Strengthening 1-randomness



Highly Nonrandom Sequences

K-triviality

If α is computable then we can describe $\alpha \upharpoonright n$ by describing n and giving an algorithm for α, which does not depend on n.

So $C(\alpha \upharpoonright n) \leqslant C(n) + O(1)$ and $K(\alpha \upharpoonright n) \leqslant K(n) + O(1)$.

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We say that α is *K-trivial* if $K(\alpha \upharpoonright n) \leqslant K(n) + O(1)$.

Other Notions of Randomness Theoretic Weakness

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α is a base for 1-randomness if there is a $\beta \geqslant_{\mathrm{T}} \alpha$ s.t. β is α-1-random (equivalently, if $\{\beta : \alpha \leqslant_{\mathrm{T}} \beta\}$ is not α-ML-null).

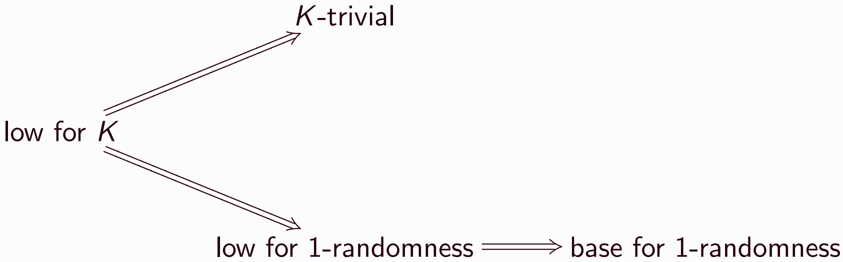
Easy Implications

α is K-trivial if $K(\alpha \restriction n) \leqslant K(n) + O(1)$.

α is low for 1-randomness if every 1-random is α-1-random.

α is low for K if $K^{\alpha}(\sigma) = K(\sigma) \pm O(1)$.

α is a base for 1-randomness if there is a $\beta \geqslant_{\mathbf{T}} \alpha$ s.t. β is α-1-random.



A Remarkable Coincidence

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Thus all four notions of randomness theoretic weakness coincide.

How Chaos Resembles Order

Highly random objects can resemble highly patterned ones.

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A musical example.

Excerpt A: >

Excerpt B: >

How Chaos Resembles Order

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A musical example.

Excerpt A: from *Music of Changes* by John Cage

Excerpt B: from *Structures for Two Pianos* by Pierre Boulez

Cage's piece is an example of aleatory music.

Boulez's piece is an example of total serialism.

How Chaos Resembles Order: Mathematical Examples

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Thm (Miller). Every 3-random sequence is weakly low for K.

Open Problem. Give a precise characterization of a notion of "useless information" that explains these and similar phenomena.

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If α is 2-random then it is $\emptyset'$-1-random, and so Ω-1-random.

By van Lambalgen's Theorem, Ω is α-1-random, so α is low for Ω. □

Surveys, Lecture Notes, etc.

homepages.mcs.vuw.ac.nz/~downey/

www.cs.auckland.ac.nz/~nies/

www.math.uchicago.edu/~drh/

www.math.dartmouth.edu/~frg/