Logic, Combinatorics and Topological Dynamics, I

Alexander S. Kechris

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Introduction

During the last few years there has been considerable activity in the study of the dynamics of automorphism groups of countable structures (in the sense of model theory). This work has led to intriguing interactions between logic, combinatorics, topological dynamics, group theory (both in the topological and algebraic context) and ergodic theory. In the last two lectures, I will give an introduction to this new area of research.

Automorphism groups

Definition

A *structure* $\boldsymbol{A} = \langle A, f, g, \ldots, R, S, \ldots, c, d, \ldots \rangle$ is a set A together with families of distinguished functions (of several variables) with arguments and values in A, relations (of several arguments) on A and individual elements of A. In these lectures, I am always assuming that there only countably many such functions, relations and individual elements.

Examples

- linear orders: $\boldsymbol{L} = \langle L, < \rangle$.
- graphs: $\boldsymbol{G} = \langle G, E \rangle$.
- groups: $\boldsymbol{H} = \langle H, \cdot, 1 \rangle$.
- vector spaces over a field F: $\boldsymbol{V} = \langle V, +, f_a \rangle_{a \in F}$.

Definition

A structure $\boldsymbol{A}$ as above is countable (resp., finite) if the set A is countable (resp., finite).

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Fraïssé structures

Certain countable structures play a crucial role in this theory.

Definition

A countable structure $\boldsymbol{K}$ is a Fraïssé structure if it satisfies the following properties:

- It is infinite.
- It is locally finite.
- It is ultrahomogeneous (i.e., an isomorphism between finite substructures can be extended to an automorphism of the whole structure).

Examples

- $\langle \mathbb{Q}, < \rangle$.
- The random graph.
- The (countable) atomless Boolean algebra.
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The age, $\mathrm{Age}(\boldsymbol{K})$, of a Fraïssé structure $\boldsymbol{K}$ is the family of its finite substructures.

Definition

A class $\mathcal{K}$ of finite structures is called a Fraïssé class if it satisfies the following properties:

- (HP) Hereditary property.
- (JEP) Joint embedding property.
- (AP) Amalgamation property.
- It is countable (up to $\cong$).
- It is unbounded.

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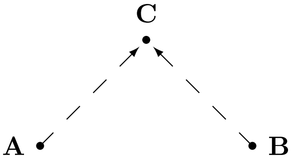

Amalgamation property (AP)

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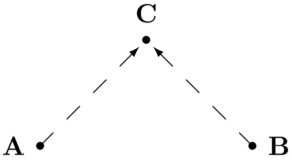

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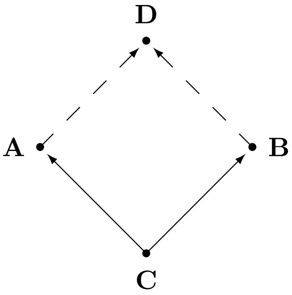

Fraïssé structures

Fraïssé showed that one can associate to each Fraïssé class $\mathcal{K}$ a canonical Fraïssé structure $\boldsymbol{K} = \mathrm{Frlim}(\mathcal{K})$, called its Fraïssé limit, which is the unique Fraïssé structure whose age is equal to $\mathcal{K}$ and therefore one has a canonical one-to-one correspondence:

$$\mathcal{K} \mapsto \mathrm{Frlim}(\mathcal{K})$$

between Fraïssé classes and Fraïssé structures whose inverse is:

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Examples

- finite graphs $\rightleftarrows$ random graph
- finite linear orderings $\rightleftarrows$ $\langle \mathbb{Q}, < \rangle$
- f.d. vector spaces $\rightleftarrows$ infinite-dimensional vector space (over a finite field)
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$\mathrm{Aut}(\boldsymbol{A})$ as a topological group

For a countable structure $\boldsymbol{A}$, we view $\mathrm{Aut}(\boldsymbol{A})$ as a topological group with the pointwise convergence topology. It is not hard to check then that it becomes a Polish group. In fact we can characterize these groups as follows:

Theorem

For any Polish group G, the following are equivalent:

- *G is isomorphic to a closed subgroup of S_∞, the permutation group of $\mathbb{N}$ with the pointwise convergence topology.*
- *G admits a countable basis at 1 consisting of open subgroups.*
- *$G \cong \mathrm{Aut}(\boldsymbol{A})$, for a countable structure $\boldsymbol{A}$.*
- *$G \cong \mathrm{Aut}(\boldsymbol{K})$, for a Fraïssé structure $\boldsymbol{K}$.*

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Dynamics of $\mathrm{Aut}(\boldsymbol{A})$

We will see how the study of the dynamics of these automorphism groups is connected with finite combinatorics, topological dynamics, group theory (topological and algebraic), and ergodic theory.

The applicability of this work also extends to the study of other important Polish groups via dense embeddings.

Example

Let $\mathcal{K}$ be the class of finite measure algebras with measure taking dyadic rational values. Its (Fraïssé) limit $\boldsymbol{K}$ is the measure algebra of clopen subsets of the Cantor space $2^{\mathbb{N}}$ with the usual product measure μ. Then there is a canonical dense embedding of the group $\mathrm{Aut}(\boldsymbol{K})$ into the group $\mathrm{Aut}(2^{\mathbb{N}}, \mu)$ of measure-preserving automorphisms, an important group in ergodic theory.

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Part I. Universal minimal flows and structural Ramsey theory

Universal minimal flows

Below G is a (Hausdorff) topological group. A G-flow is a continuous action of G on a (Hausdorff, nonempty) compact space X. A subflow of X is a compact invariant set with the restriction of the action. A flow is minimal if there are no proper subflows or equivalently every orbit is dense. Every G-flow contains a minimal subflow. A homomorphism between two G-flows X, Y is a continuous G-map $\pi : X \to Y$. If Y is minimal, then π must be onto. An isomorphism is a bijective homomorphism.

Theorem

For any G, there is a minimal G-flow, $M(G)$, with the following property: For any minimal G-flow X, there is a homomorphism $\pi : M(G) \to X$. Moreover $M(G)$ is uniquely determined up to isomorphism by this property.

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Universal minimal flows

If G is compact, then $M(G) = G$. If G is non-compact but locally compact, then $M(G)$ is very big, e.g., it is non-metrizable. However, it is a remarkable phenomenon that for non-locally compact groups G, $M(G)$ can even trivialize (i.e., can be a singleton)!

This leads to two general problems in topological dynamics:

- When is $M(G)$ trivial?
- Even if it is not trivial, can one explicitly determine $M(G)$ and show that it is manageable, in particular metrizable?

There has been an extensive study of these problems in the last 20 years or so in the work of Gromov, Milman, Glasner, Weiss, Giordano, Pestov, Uspenskii and others.

This primarily involves two ingredients:

- asymptotic geometric analysis (concentration of measure phenomena): Gromov, Milman, Pestov.
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Extreme amenability

Definition

A group G is called extremely amenable if its universal minimal flow $M(G)$ is trivial.

This is equivalent to saying that G has an extremely strong fixed point property: Every G-flow has a fixed point. For that reason, sometimes extremely amenable groups are also said to have the fixed point on compacta property.

T. Mitchell (1966) raised the question of their existence. Granirer-Lau and Veech showed in the 1970's that no locally compact group can be extremely amenable. The first examples of extremely amenable groups were produced by Herer-Christensen (1975), who, apparently unaware of Mitchell's question, showed that there are Polish abelian groups that are "exotic", i.e., admit no non-trivial unitary representations. Such groups are extremely amenable.

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The first natural example of an extremely amenable group was produced by Gromov-Milman (1983): $U(H)$. The proof used concentration of measure techniques. By such methods other important examples were discovered later:

- Furstenberg-Weiss, Glasner (1998): $L(X, \mu, \mathbb{T})$.
- Pestov (2002): $\mathrm{Iso}(\mathbb{U})$.
- Giordano-Pestov (2002): $\mathrm{Aut}(X, \mu)$.

Pestov (1998) also produced another example: $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$. His proof however did not use concentration of measure techniques but rather finite combinatorics, more specifically the classical Ramsey Theorem. From this it also follows that $H_+([0,1])$ is extremely amenable.

Extreme amenability

The first natural example of an extremely amenable group was produced by Gromov-Milman (1983): $U(H)$. The proof used concentration of measure techniques. By such methods other important examples were discovered later:

- Furstenberg-Weiss, Glasner (1998): $L(X, \mu, \mathbb{T})$.
- Pestov (2002): $\mathrm{Iso}(\mathbb{U})$.
- Giordano-Pestov (2002): $\mathrm{Aut}(X, \mu)$.

Pestov (1998) also produced another example: $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$. His proof however did not use concentration of measure techniques but rather finite combinatorics, more specifically the classical Ramsey Theorem. From this it also follows that $H_+([0,1])$ is extremely amenable.

Metrizable universal minimal flows

The first example of calculation of a metrizable but non-trivial universal minimal flow is due to Pestov (1998): The universal minimal flow of $H_+(\mathbb{T})$ is $\mathbb{T}$. Two more examples were found later by Glasner-Weiss (2002,2003): The universal minimal flow of S_∞ is the space LO of linear orderings of $\mathbb{N}$. The universal minimal flow of $H(2^{\mathbb{N}})$ is the Uspenskii space of maximal chains of closed subsets of the Cantor space. These all used Ramsey techniques.

Universal minimal flows of automorphism groups

We will next discuss the study of extreme amenability and calculation of universal minimal flows for automorphism groups of countable structures. This was undertaken in a paper of K-Pestov-Todorcevic (GAFA, 2005). The main result of this theory is the development of a duality theory which shows that there is an equivalence between the structure of the universal minimal flow of the automorphism group of a Fraïssé structure and the Ramsey theory of its finite "approximations", i.e., its age.

Structural Ramsey theory

We first recall the classical Ramsey Theorem.

Theorem (Ramsey 1930)

For each $n, m, k \geq 1$, with $m \geq k$, there is $M \geq m$, such that if we color the k-element subsets of $\{1, \ldots, M\}$ with n colors, there is a subset X of $\{1, \ldots, M\}$ of size m which is monochromatic, i.e., all k-element subsets of X have the same color.

We abbreviate by:

$$M \longrightarrow (m)^k_n$$

this last assertion, so that Ramsey's theorem says that: For each $n, m, k \geq 1$, with $m \geq k$, there is $M \geq m$ such that $M \longrightarrow (m)^k_n$.

Equivalent formulation: For n, m, k as above, there is M such that if we color all increasing k-tuples in $\{1, \ldots, M\}$ with n colors, there is an increasing m-tuple X in $\{1, \ldots, M\}$, such that all increasing k-tuples from X have the same color.

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Definition

A class $\mathcal{K}$ of finite structures has the Ramsey property if for any $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$, and any $n \geq 1$, there is $\boldsymbol{C} \geq \boldsymbol{B}$ in $\mathcal{K}$, such that

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_n.$$

Examples of classes with Ramsey property:

- finite linear orderings (Ramsey)
- finite Boolean algebras (Graham-Rothchild)
- finite-dimensional vector spaces over a given finite field (Graham-Leeb-Rothchild)
- finite ordered graphs (Nešetřil-Rödl)

However, the class of finite graphs does not have the Ramsey property!

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We can now summarize in general terms the main point of the duality theory alluded to earlier:

Let $\mathcal{K}$ be a Fraïssé class of finite structures and $\boldsymbol{K} = \mathrm{Frlim}(\mathcal{K})$ its Fraïssé limit. Then we have a canonical correspondence:

structure of the u.m.f. of $\mathrm{Aut}(\boldsymbol{K})$ $\leftrightarrow$ Ramsey theory of $\mathcal{K}$

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Extreme amenability of automorphism groups

We will first consider the problem of characterizing the extremely amenable closed subgroups G of S_∞. We have seen they are all of the form $G = \mathrm{Aut}(\boldsymbol{K})$ for a Fraïssé structure $\boldsymbol{K}$. If G is e.a., look at its action on LO. It leaves some order $<$ invariant, so we can assume that $\boldsymbol{K} = \langle K, <, \dots \rangle$ is an ordered Fraïssé structure. So $\boldsymbol{K} = \mathrm{Frlim}(\mathcal{K})$, for an order Fraïssé class $\mathcal{K}$, i.e., each $\boldsymbol{A}$ has the form $\boldsymbol{A} = \langle A, <, \dots \rangle$.

Examples

- finite ordered graphs
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So we have seen that the extremely amenable closed subgroups of S_∞ are to be found among the $\mathrm{Aut}(\boldsymbol{K})$, where $\boldsymbol{K}$ is the limit of an order Fraïssé class $\mathcal{K}$. But which ones?

Theorem (KPT)

Let $\mathcal{K}$ be an order Fraïssé class and $\boldsymbol{K}$ its limit. Then the following are equivalent:

- $\mathrm{Aut}(\boldsymbol{K})$ *is extremely amenable.*
- $\mathcal{K}$ *has the Ramsey property.*

Using the results of the structural Ramsey theory gives now a plethora of new examples of interesting extremely amenable groups.

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Extreme amenability of automorphism groups

Below we consider only Fraïssé order classes.

Fraïssé classes of finite structures $\mathcal{K}$	Fraïssé structures $\boldsymbol{K}$
Ramsey property of $\mathcal{K}$	extreme amenability of $\mathrm{Aut}(\boldsymbol{K})$
linear orders	$\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$
ordered graphs	$\mathrm{Aut}(\langle \boldsymbol{R}, < \rangle)$
lex. ordered vector spaces	$\mathrm{Aut}(\langle \boldsymbol{V}_\infty, < \rangle)$
lex. ordered Boolean algebras	$\mathrm{Aut}(\langle \boldsymbol{B}_\infty, < \rangle)$
ordered rational metric spaces	$\mathrm{Aut}(\langle \boldsymbol{U}_\mathbb{Q}, < \rangle)$

Calculation of universal minimal flows

This duality theory also extends to the calculation of metrizable minimal flows for automorphism groups. Roughly speaking one can assign to each Fraïssé class $\mathcal{K}$ with limit $\boldsymbol{K} = \langle K, \dots \rangle$ certain canonical expansions $\mathcal{L}$ consisting of structures of the form $\langle \boldsymbol{A}, < \rangle$, obtained by adding to each structure in $\mathcal{K}$ appropriate "canonical orderings", and for each such $\mathcal{L}$ a canonical flow $X_{\mathcal{L}}$ of the automorphism group of $\boldsymbol{K}$, which is a compact metrizable space of "canonical orderings" on K. Then we have the following:

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- $\mathcal{K}$ = finite graphs, $\boldsymbol{K} = \boldsymbol{R}$; $\mathcal{L}$ = finite ordered graphs. Then $X_{\mathcal{L}}$ is the space of all linear orderings on the vertices of the random graph.
- $\mathcal{K}$ = finite sets, $\boldsymbol{K} = \langle \mathbb{N} \rangle$; $\mathcal{L}$ = finite orderings. Then $X_{\mathcal{L}}$ is the space of all linear orderings on $\mathbb{N}$ (Glasner-Weiss).
- $\mathcal{K}$ = f.d. vector spaces over a fixed finite field, $\boldsymbol{K} = \boldsymbol{V}_{\infty}$; $\mathcal{L}$ = lex. ordered f.d. vector spaces. Then $X_{\mathcal{L}}$ is the space of all "canonical orderings" on $\boldsymbol{V}_{\infty}$.
- $\mathcal{K}$ = finite posets, $\boldsymbol{K} = \boldsymbol{P}$; $\mathcal{L}$ = finite posets with linear extensions. Then $X_{\mathcal{L}}$ is the space of all linear extensions of the random poset.

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Summary of applications of the duality theory

- Establishes the equivalence between the structure of the universal minimal flow of the automorphism group of the limit of a Fraïssé class and its Ramsey properties and therefore can use the extensive structural Ramsey theory to analyze such universal minimal flows and discover many new examples of extremely amenable groups. This application goes from Ramsey theory to topological dynamics. The following is an interesting question:

  Can one go in the other direction: use topological dynamics methods to prove Ramsey theorems?

  Only a couple of rather simple results are known in this direction.

  However this duality has had an interesting indirect effect on structural Ramsey theory. In trying to applying duality theory to various Fraïssé classes that occur naturally, it led to the discovery of new structural Ramsey theorems:

  1. (Nešetřil, 2007) finite ordered metric spaces. (Nguyen Van The, 2008) other classes of finite ordered metric spaces.
  2. (K-Todorcevic) lex. ordered finite measure algebras (with dyadic rational measure).

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- Establishes the equivalence between the structure of the universal minimal flow of the automorphism group of the limit of a Fraïssé class and its Ramsey properties and therefore can use the extensive structural Ramsey theory to analyze such universal minimal flows and discover many new examples of extremely amenable groups. This application goes from Ramsey theory to topological dynamics. The following is an interesting question:

  Can one go in the other direction: use topological dynamics methods to prove Ramsey theorems?

  Only a couple of rather simple results are known in this direction.

  However this duality has had an interesting indirect effect on structural Ramsey theory. In trying to applying duality theory to various Fraïssé classes that occur naturally, it led to the discovery of new structural Ramsey theorems:

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- Automorphism groups of Fraïssé structures often admit dense embeddings into other "larger" Polish groups. If G is extremely amenable and can be densely embedded in H, then H is also extremely amenable. Thus results concerning extreme amenability of automorphism groups, which use combinatorial methods, can be used to establish extreme amenability of other groups which were originally established by concentration of measure techniques (which, by the way, fail in the context of automorphism groups). Here are some examples:
  1. The isometry group of the Urysohn space (originally proved by Pestov).
  2. The automorphism group of a standard measure space (originally established by Giordano-Pestov).
  3. Question: Can that be done for the unitary group?
- Connection with infinite Ramsey theory, distortion phenomena in Banach spaces.

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