Logic, Combinatorics and Topological Dynamics, II

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Part II. Generic symmetries

Generic properties

Let X be a topological space and $P \subseteq X$ a subset of X viewed as a property of elements of X. As usual, we say that P is generic if it is comeager in X.

Example

Nowhere differentiability is a generic property in $C([0,1])$.

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Generic elements

Suppose now X is a topological space equipped with a natural equivalence relation E. Then we view the relation xEy as "identifying" in some sense x and y.

We then say that $x_0 \in X$ is generic (relative to E) if the E-equivalence class of x is comeager, i.e., the generic element of X is E-equivalent, i.e., "identical", to x_0.

Examples

- Suppose a topological group G acts on X. Then an element $x_0 \in X$ is generic (for this action) if its orbit $G \cdot x_0$ is comeager.
- Consider the particular case when a topological group G acts on itself by conjugation. Then $g_0 \in G$ is generic if its conjugacy class is comeager.

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Generic symmetries

Most often G is the group of symmetries of a mathematical structure.

Examples

- $H(X)$
- $U(H)$
- $\mathrm{Iso}(X, d)$
- $\mathrm{Aut}(X, \mu)$
- $\mathrm{Aut}(\boldsymbol{K})$

Then we talk about a generic symmetry of the structure.

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Generic symmetries

When does G have generic elements?

We have here a vague dichotomy.

- "big groups", like $U(H)$, $\mathrm{Aut}(X, \mu), \ldots$ do *not* have generic elements.
- "small groups", like $\mathrm{Aut}(\boldsymbol{K})$, $\boldsymbol{K}$ a countable structure, often have generic elements.

I will describe now recent work of K-Rosendal (2007) that studies the problem of generic automorphisms of Fraïssé structures and its implications.

This kind of problem was first studied by Lascar, Truss in model theory.

It also arose in topological dynamics, e.g., in work of Akin-Hurley-Kennedy (Memoirs of the AMS, 2003), who studied generic properties of homeomorphisms of the Cantor space and asked whether it has a generic homeomorphism. By Stone duality this is of course the same as asking whether there is a generic automorphism of $\boldsymbol{B}_\infty$.

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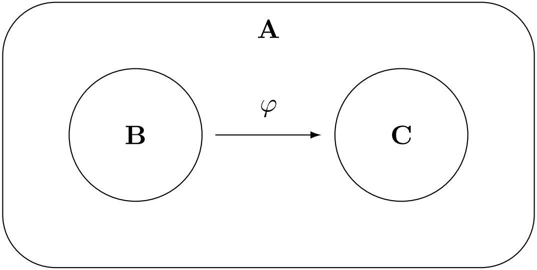
Generic automorphisms of Fraïssé structures

Let $\mathcal{K}$ be a Fraïssé class of finite structures and $\boldsymbol{K} = \mathrm{Frlim}(\mathcal{K})$ its limit. Truss has associated to $\mathcal{K}$ a new class of finite objects $\mathcal{K}_p$ consisting of all pairs

$$(\boldsymbol{A}, \varphi : \boldsymbol{B} \to \boldsymbol{C}),$$

where $\boldsymbol{B}, \boldsymbol{C} \subseteq \boldsymbol{A} \in \mathcal{K}$ and φ is an isomorphism of $\boldsymbol{B}, \boldsymbol{C}$.



Truss found a sufficient condition for the existence of generic automorphisms in terms of properties of $\mathcal{K}_p$.

Theorem (Truss)

If a cofinal class in $\mathcal{K}_p$ has the JEP and the AP, then there is a generic automorphism of $\boldsymbol{K}$.

Truss also asked for necessary and sufficient conditions.

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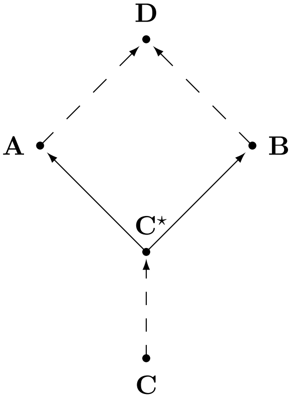

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Theorem (KR)

The structure $\boldsymbol{K}$ has a generic automorphism iff $\mathcal{K}_p$ has the JEP and the WAP.

This result was also proved by Ivanov (1999) for $\aleph_0$-categorical structures using different techniques.

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Generic automorphisms of Fraïssé structures

We can now apply these to show the existence of generic automorphisms for many Fraïssé structures:

- (Truss, Kuske-Truss) The groups S_∞, $\mathrm{Aut}(\boldsymbol{P})$, $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ have generic automorphisms.
- (KR) The countable atomless Boolean algebra has a generic automorphism and thus the Cantor space has a generic homeomorphism.
- More examples below ...

Very recently, Akin-Glasner-Weiss found another, topological, proof of the existence of generic homeomorphisms of the Cantor space and gave a characterization of its properties.

Note: There are Polish groups not contained in S_∞ that have generic elements, e.g., the group of increasing homeomorphisms of the interval [0,1].

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Ample generics

We will now discuss a multidimensional notion of genericity.

Definition

Let a group G act on a topological space X. Then G also acts on X^n coordinatewise

$$g \cdot (x_1, \ldots, x_n) = (g \cdot x_1, \ldots, g \cdot x_n).$$

We say that $(x_1, \ldots, x_n)$ is generic if it is generic for this action, i.e., its orbit is comeager. We finally say that the action of G on X has ample generics if for each n, there is a generic element of X^n. Applying this to the conjugacy action of a topological group on itself, we say that G has *ample generics* if for each n, there is $(g_1, \ldots, g_n)$ such that

$$\{(gg_1g^{-1}, \ldots, gg_ng^{-1}) : g \in G\}$$

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There are now many examples of countable structures whose automorphism groups are known to have ample generics.

Examples

1. S_∞
2. $\mathrm{Aut}(\boldsymbol{R})$ (Hrushovski)
3. Many automorphism groups of ω-stable, $\aleph_0$-categorical structures (Hodges-Hodkinson-Lascar-Shelah)
4. $\mathrm{Aut}(\mathbb{U}_0)$ (Solecki, Vershik)
5. $\mathrm{Aut}(F_\infty)$ (Bryant-Evans)
6. The group of measure-preserving homeomorphisms of the Cantor space (KR)
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Ample generics

There are however structures whose automorphism groups have generic elements but not ample generics.

Theorem (Hodkinson)

The automorphism group of $\langle \mathbb{Q}, < \rangle$ has generic elements but not ample generics.

The following is a very interesting open problem:

Does the automorphism group of the countable atomless Boolean algebra have ample generics? (It does have generic elements.) Equivalently does the homeomorphism group of the Cantor space have ample generics?

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Does the automorphism group of the countable atomless Boolean algebra have ample generics? (It does have generic elements.) Equivalently does the homeomorphism group of the Cantor space have ample generics?

Another important open problem is to find examples of Polish groups with ample generics that are not closed subgroups of S_∞.

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Another important open problem is to find examples of Polish groups with ample generics that are not closed subgroups of S_∞.

Ample generics

It turns out that Polish groups with ample generics have remarkable properties and I will discuss these in the rest of this lecture.

The small index property

Definition

A Polish group has the small index property (SIP) if every subgroup of index less than $2^{\aleph_0}$ is open.

Thus for closed subgroups G of S_∞, SIP implies that the topology of G is determined by its algebra.

Hodges-Hodkinson-Lascar-Shelah used (special types of) ample generics to prove SIP for the automorphism groups of certain structures. It turns out that this is a general phenomenon.

Theorem (KR)

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Automatic continuity

Automatic continuity results are known for certain types of algebras but the next result appears to produce the first instance of this phenomenon, in a general framework, in the context of groups.

Theorem (KR)

If a Polish group G has ample generics, then any (algebraic) homomorphism of G into a separable group is continuous.

In particular, such groups have a unique Polish (group) topology. For S_∞ one has actually a stronger result.

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The group S_∞ has a unique non-trivial separable (group) topology.

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This automatic continuity phenomenon has been extended by Rosendal-Solecki to other kinds of groups that are only known to admit (single) generics. These include:

- $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$
- $H(2^{\mathbb{N}})$
- $H_+(\mathbb{R})$

In particular, this implies that $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ as a discrete group is extremely amenable relative to compact metric spaces!

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Uncountable cofinality

Definition

A group G has uncountable cofinality if it cannot be written as a union of an increasing sequence of proper subgroups.

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If a Polish group has ample generics, then G cannot be written as a union of an increasing sequence of non-open subgroups.

So, for example, if such a group is either connected or topologically finitely generated or an oligomorphic closed subgroup of S_∞, then it has uncountable cofinality.

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The Bergman property, isometric actions and actions on trees

Definition

A group G has the Bergman property if for any sequence $E_n \subseteq E_{n+1} \subseteq G$ with $G = \bigcup_n E_n$, there is some n, k with $G = (E_n)^k$.

Each of the following are equivalent descriptions of the Bergman property:

1. (a) For every symmetric generating set S of G containing 1, there is n with $G = S^n$ and (b) G has uncountable cofinality.
2. Every action of G by isometries on a metric space has bounded orbits.

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Bergman (2004) introduced this property and proved that S_∞ has the Bergman property.

Theorem (KR)

If G is an oligomorphic closed subgroup of S_∞ with ample generics, then G has the Bergman property.

Recently, de Cornulier, using results of Calegari and Freedman, proved that the homeomorphism group of S^n has the Bergman property. Also Ricard and Rosendal showed that the unitary group of the separable infinite-dimensional Hilbert space has the Bergman property. Finally, B. Miller showed that $\mathrm{Aut}(X, \mu)$ has the Bergman property.

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The Bergman property, isometric actions and actions on trees

Definition

A group G has Serre's property (FA) if any action of G on a tree has a fixed vertex or edge.

Theorem (KR)

If a Polish group is either connected or topologically finitely generated or an oligomorphic closed subgroup of S_∞ and has ample generics, then G has property (FA).

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Some problems

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Are there Polish groups with ample generics that are not closed subgroups of S_∞?

Problem

Are there Polish locally compact groups with generics?

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Does the homeomorphism group of the Cantor space have ample generics?

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