Randomness for Continuous Measures 4

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Higher Orders of Randomness

necessity of the set theoretic methods

Theorem

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We are discussing the proof for $n = 0$.

Gödel's L

Definition

Gödel's hierarchy of constructible sets L is defined by the following recursion.

- $L_0 = \emptyset$
- $L_{\alpha+1} = Def(L_\alpha)$, the set of subsets of L_α which are first order definable in parameters over L_α.
- $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.

We focus on the least ordinal λ such that $L_\lambda \models ZFC^-$.

Master Codes

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We show that there is an n such that the Master Codes in L_λ are in NCR_n.

Pseudo Master Code Sequences

For any Z, Z can arithmetically define a sequence $\mathcal{M}$ consisting of a linear ordering $<$ of pseudo-master-codes.

- If M is in the well-founded part of $\mathcal{M}$, then M is a master-code.
- If M is not in the well-founded part of $\mathcal{M}$, then M is not well-founded.

Finding n so that the Master Codes belong to NCR_n

Choose n much larger than the propagating definitions of Master Codes and much larger than the definition mapping a real Z to its sequence $\mathcal{M}$.

Let M_β be a Master Code and suppose that M_β is n-random relative to μ.

Finishing

1. Fix the sequence $\mathcal{M}$ for μ.
   - The well-founded part $\mathcal{M}_0$ of $\mathcal{M}$ must have length less than or equal to β.
   - Then, it is recursive in M_β and hence simply arithmetic in μ. (Randoms do not accelerate computing well-foundedness.)

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2. Let γ be least Master Code not in $\mathcal{M}_0$. But M_γ is simply arithmetic in $\mathcal{M}_0$ and recursive in M_β. Hence M_γ is recursive in μ. (Randoms do not accelerate computing simple arithmetic sets.)

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3. Contradiction!
4. So $M_\beta \in NCR_n$.

Understanding NCR_n

Definition

A Π^1_1-subset P of 2^ω is one with a definition of the form $X \in P \iff (\forall Y \in \omega^\omega)(\exists n)\varphi(n, X \upharpoonright n, Y \upharpoonright n)$, in which φ has only bounded quantifiers.

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NCR_n has this form: $X \in NCR_n$ if and only if for all $m \in 2^\omega$, m does not represent a continuous measure for which X is k-random.

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For each n, NCR_n is a countable Π^1_1-subset of 2^ω which is not Δ^1_1.

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For each n, NCR_n is a countable Π^1_1-subset of 2^ω which is not Δ^1_1.

We should understand NCR_n as a Π^1_1-set.

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- Let T_X be the set of $\tau \in \omega^\omega$, such that for all n less than the length of τ, $\neg\varphi(n, X \upharpoonright n, \tau \upharpoonright n)$.

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- Let T_X be the set of $\tau \in \omega^\omega$, such that for all n less than the length of τ, $\neg\varphi(n, X \upharpoonright n, \tau \upharpoonright n)$.
- Then $X \in P$ if and only if T_X is well-founded, if and only if T_X has an ordinal rank.

Understanding NCR_n

Question

Is there a natural Π^1_1-norm (ordinal ranking) of NCR_n, explaining the connection between definability (constructibility) and failure of continuous randomness?

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Is there a natural Π^1_1-norm (ordinal ranking) of NCR_n, explaining the connection between definability (constructibility) and failure of continuous randomness?

This question is open, though we have some information about the special case NCR_1.

NCR_1

- Kjos-Hanssen and Montalbán showed that the elements of countable Π^0_1 classes belong to NCR_1.

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Theorem

There is an X in NCR_1 which is not in any countable Π^0_1-class.

Other Examples

Theorem

NCR_1 contains elements of the following types.

- *1-generic*
- *packing dimension 1*
- *recursively enumerable*
- *minimal*

Theorem (Montalbán and Slaman)

NCR_1 contains all K-trivial sequences.

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Question

Is $NCR_1 \cap \Delta^0_3$ hyperarithmetically definable?

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