Randomness for Continuous Measures 3

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Features of the proof:

- Applies Martin's theorem that all arithmetic games on 2^ω are determined.
- Concludes that the elements of NCR_k are definable. They belong to the least initial segment of Godel's universe of constructible sets L_α such that

$$L_\alpha \models ZFC_k^-,$$

where ZFC_k^- is Zermelo-Frankel set theory with only k iterates of the power set of ω.

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Theorem (H. Friedman 1971)

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Essentially, we will show that the determinacy relatively-random game requires iterations of the power set of ω.

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- If Y is recursive in $X \oplus \mu$ and not recursive in μ, then Y is $(n-2)$-random for some continuous measure μ_Y recursive in μ'' (relative to μ''). (Apply a theorem of Demuth.)

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In general, using arithmetic definitions with fewer than n quantifiers, n-random reals do not accelerate arithmetic definability and nontrivially define only relatively random reals.

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Theorem

Suppose that X is 5-random relative to μ, $\prec$ is recursive in μ, and I is the largest initial segment of $\prec$ which is well-founded. If I is recursive in $X \oplus \mu$, then I is recursive in μ.

Proof

Suppose $I \leq_T X \oplus \mu$ and $I \not\leq_T \mu$. Then, there is a continuous μ_I recursive in μ'' such that I is 3-random for μ_I relative to μ''.

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For $\alpha \in \omega$, let $\mathcal{I}(\alpha)$ be the set of X's such that X is an initial segment of $\prec$ and all of X's elements are bounded by α. Note that $\mathcal{I}(\alpha)$ is $\Pi^0_1(\mu)$. Hence, there is a μ''-effective procedure to tell whether $\mathcal{I}(\alpha)$ has positive μ_I-measure.

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- If $\alpha \in I$, then $\mathcal{I}(\alpha)$ is countable and $\mu_I(\mathcal{I}(\alpha)) = 0$.
- If $\alpha \notin I$, then $I \in \mathcal{I}(\alpha)$, I is 3-random for μ_I relative to μ'', and so $\mu_I(\mathcal{I}(\alpha)) \neq 0$.

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- If $\alpha \in I$, then $\mathcal{I}(\alpha)$ is countable and $\mu_I(\mathcal{I}(\alpha)) = 0$.
- If $\alpha \notin I$, then $I \in \mathcal{I}(\alpha)$, I is 3-random for μ_I relative to μ'', and so $\mu_I(\mathcal{I}(\alpha)) \neq 0$.

Thus, I is Π^0_2 relative to μ'', contradiction to I's being 3-random for μ_I relative to μ''.

Higher Orders of Randomness

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We will sketch the proof for $n = 0$ and indicate how to adapt it for $n > 0$.

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- Say $0^{(k)}$ is 3-random relative to μ.
- $0'$ is recursively enumerable relative to μ and recursive in the supposedly 3-random $0^{(k)}$. Hence, $0'$ is recursive in μ and so $0''$ is recursively enumerable relative to μ.

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- Use induction to conclude $0^{(k)}$ is recursive in μ, a contradiction.

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- Then 0^ω is recursive in μ'', proof sketch on next slide.

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Proof

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- By the previous argument, for every k, $0^{(k)}$ is recursive in μ.
- Then 0^ω is recursive in μ'', proof sketch on next slide.
- Consequently, 0^ω is not 3-random relative to μ.

Elaboration

Proposition

Suppose for every k, $0^{(k)}$ is recursive in X. Then 0^{ω} is recursive in X''.

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$0'$ satisfies,

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Thus, $0'$ is the unique set satisfying a Π^0_2 property.

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Thus, $0'$ is the unique set satisfying a Π^0_2 property.

If $X \geq_T 0'$, then X'' can uniformly pick out the way to compute $0'$ from X. The rest follows by induction.

Gödel's L

Definition

Gödel's hierarchy of constructible sets L is defined by the following recursion.

- $L_0 = \emptyset$
- $L_{\alpha+1} = Def(L_\alpha)$, the set of subsets of L_α which are first order definable in parameters over L_α.
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- $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.

We focus on the least ordinal λ such that $L_\lambda \models ZFC^-$. We show that there is an n such that NCR_n is cofinal in the Turing degrees of L_λ.

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- For any $\beta < \lambda$ with $\beta \in LOR$, there is an $X \subset \omega$ such that $X \in Def(L_\beta) \setminus L_\beta$.
- (Putnam and Enderton) For any $\beta < \lambda$ with $\beta \in LOR$, there is an $E \subset \omega \times \omega$ such that $E \in L_{\beta+3}$ and (ω, E) is isomorphic to (L_β, ϵ). E is obtained by observing that Gödel's Condensation Theorem implies that L_β is the Skolem hull of the parameters which define the previous X in L_β.

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- (Jensen) For any $\beta < \lambda$ with $\beta \in LOR$, there is a canonical set $M_\beta \in L_{\beta+3} \cap 2^\omega$, called the *master code* for L_β, such that M_β is the elementary diagram of a canonical counting of L_β.

About the Master Codes

If $\alpha < \beta < \lambda$ and $\alpha, \beta \in LOR$, then all of X, Y, M_α, and the isomorphism between L_α and M_α's representation of L_α mentioned earlier are elements of L_β.

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For every $X \in 2^\omega \cap L_\lambda$, there is a $\beta \in LOR$ such that $\beta < \lambda$ and X is recursive in some M_β. Hence, the set $\{M_\beta : \beta < \lambda\}$ is not countable in L_λ.

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We will show that there is an n such that

$$\{M_\beta : \beta < \lambda\} \subset NCR_n.$$

About the Master Codes

obtaining M_β by iterated relative definability.

In the previous frames, we defined L by iterating first order definability from parameters and taking unions. This iteration is reflected by the master codes.

- For $\alpha \in LOR$, $M_{\alpha+\omega}$ can be defined from M_α by iterating Σ^0_1-relative definability and taking uniformly arithmetic limits.
- For a limit $\gamma \in LOR$, M_γ can be defined from the sequence of smaller M_α's by taking a uniformly arithmetic limit and then iterating Σ^0_1-relative definability.

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- For every M and N satisfying φ, either one belongs to the structure coded by the other and embeds its coded structure as an initial segment of the other's, or there is a $\Pi^0_3(M \oplus N)$ set which exhibits a failure of well-foundedness in one of their coded structures.

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For any Z there is an arithmetic φ specifying a collection of pseudo-master codes among the sets recursive in Z and an arithmetic method to linearly order the apparently well-founded models they code. Moreover, the arithemtic formulas do not depend on Z, and the well-founded codes form an initial segment of the ordering.

To be continued