Randomness for Continuous Measures 2

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Question

For which sequences $X \in 2^\omega$ *do there exist (representations of) continuous probability measures* μ *such that* X *is random for* μ*?*

Failures of Continuous Randomness

Theorem (Kjos-Hanssen and Montalbán)

Suppose that P is a countable Π^0_1-class and $X \in P$. Then there is no continuous μ such that X is 1-μ-random.

Definition

$X \in NCR_k$ if and only if there is no representation m of a continuous measure μ such that X is k-random relative to the representation m of μ.

By Kjos-Hanssen and Montalbán, every element of a countable Π^0_1-class belongs to NCR_1.

The Hyperarithmetic Sets

Definition

Suppose that $<$ is a linear ordering of ω. A jump hierarchy along $<$ is a function J from ω to 2^ω such that

- If m is the immediate successor of n in $<$, then $J(m) = J(n)'$.
- If l is a limit in $<$, then $J(l) = \{2^n 3^i : n < l \text{ and } i \in J(n)\}$.

If $<$ is a well-ordering, then there is a unique jump hierarchy along $<$.

Definition

A set X is hyperarithmetic iff there is a recursive well-ordering of ω with jump hierarchy J and an n such that $X \leq_T J(n)$.

Theorem (Kreisel, 1959)

- *If X is an element of a countable Π^0_1 set, then X is hyperarithmetic.*
- *Every hyperarithmetic Y is recursive in some H which is an element of a countable Π^0_1 set.*

Consequently, the Turing degrees of the elements of NCR_1 are cofinal in the Turing degrees of the hyperarithmetic sets.

Corollary

There is a set X such that the following conditions hold

- *X is not hyperarithmetic.*
- *There is no continuous μ with a representation m hyperarithmetic in X such that X is 1-μ-random relative to m.*

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Proof

Take a nonstandard version of one of Kreisel's P's and H's.

Higher orders of randomness

basic observations

Fact (Well-known)

Suppose that $k > 1$ and X is k-random for μ.

- *μ' is not recursive in X.*
- *Every function recursive in X is dominated by a function recursive in μ'.*

Higher orders of randomness

basic observations

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Suppose that $k > 1$ and X is k-random for μ.

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Corollary

If $k \geq 1$, R is $(k+1)$-random relative to Z, and $X \equiv_{T,Z} R$, then $X \equiv_{tt,Z'} R$. Hence, X is k-random for some continuous measure.

Higher orders of randomness

NCR_k

Theorem

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We will show that every element of NCR_k is definable. However, as k increases the envelope of definability increases dramatically.

Higher orders of randomness

a cone of Turing degrees disjoint from NCR_k

Lemma

There is a $B \in 2^\omega$*, such that* $X \geq_T B$ *implies* $X \notin NCR_k$.

Higher orders of randomness

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Lemma

There is a $B \in 2^\omega$, such that $X \geq_T B$ implies $X \notin NCR_k$.

Proof

A Borel subset of $\neg NCR_k$. Suppose $Z \in 2^\omega$, R is $(k+1)$-random relative to Z, and $X \equiv_T R \oplus Z$. Then, $X \equiv_{tt,Z'} R$, R is k-random relative to Z', and so X is k-random relative to some continuous measure.

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$\neg NCR_k$ contains a cone in $\mathcal{D}$. By the above, $\neg NCR_k$ contains the cofinal and degree-invariant set

$$\{Y : \exists Z \exists R (R \text{ is 3-random in } Z \text{ and } Y \equiv_T Z \oplus R).\}$$

This set is clearly cofinal in $\mathcal{D}$. By Borel Determinacy, it contains a cone in $\mathcal{D}$.

Higher orders of randomness

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- Martin's proof of Borel Determinacy starts with a description of a Borel game and produces a winning strategy for one of the players.
- The more complicated the game is in the Borel hierarchy, the more iterates of the power set of the continuum are used in producing the strategy.
- The absoluteness of Π^1_1 sentences between well-founded models and the direct nature of Martin's proof imply that if G is a real parameter used to define a Borel game, then the winning strategy for that game belongs to the smallest $L_\beta[G]$ such that $L_\beta[G]$ is a model of a sufficiently large subset of ZFC.

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We will work with models of ZFC^-_k, which is ZFC with only k iterates of the power set of ω. Let L_β be the smallest well-founded model of ZFC^-_k. Note, L_β is countable.

Higher orders of randomness

a join theorem

Lemma

Suppose that $X \notin L_\beta$. Then there is a G such that

- *$L_\beta[G]$ is a model of ZFC_k^-.*
- *Every element of $2^\omega \bigcap L_\beta[G]$ is recursive in $X \oplus G$.*

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Proof

Use Kumabe–Slaman forcing P to generically extend L_β. This forcing builds a functional Φ_G by finite approximation.

- The elements p of the forcing partial order P are pairs $(\Phi_p, \overrightarrow{X}_p)$ in which Φ_p is a finite use-monotone functional and $\overrightarrow{X}_p$ is a finite subset of 2^ω.
- If p and q are elements of P, then $p \geq q$ if and only if
  - $\Phi_p \subseteq \Phi_q$ and for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,
  - $\overrightarrow{X}_p \subseteq \overrightarrow{X}_q$,
  - for every x, y, and $X \in \overrightarrow{X}_p$, if $\Phi_q(x, X) = y$ then $\Phi_p(x, X) = y$.

Higher orders of randomness

a join theorem

The definability of forcing and compactness show that if $D \in L_\beta$ is dense and $p \in P$, then there is a q in D extending p such that q makes no additional commitments about $\Phi_G(X)$.

Thus, for each term τ in the forcing language and each $n \in \omega$, it is possible to decide $n \in \tau$ and then extend our commitment on $\Phi_G(X)$ to record this decision.

We construct G in ω-many steps so that G is P-generic for L_β and so that $\Phi_G(X)$ records what is forced during our construction. $\square$

Higher orders of randomness

$NCR_k \subseteq L_\beta$.

Corollary

$NCR_k \subseteq L_\beta$. Hence, NCR_k is countable.

Proof

Suppose $X \notin L_\beta$ and apply the previous lemma to obtain a G such that $L_\beta[G]$ is a model of ZFC^-_k and every element of $2^\omega \bigcap L_\beta[G]$ is recursive in $X \oplus G$.

Relative to G, X belongs to every cone with base in $L_\beta[G]$. By a quantifier count for the randomness game, X belongs to the cone avoiding NCR_k relative to G.

Thus, there is a continuous measure μ such that X is k-random for μ relative to G.

But then, X is k-random for a continuous μ, as required.

Higher orders of randomness

obtaining μ from X

Given $X \notin L_\beta$, we showed that there is a continuous μ such that X is k-random for μ. We can define such a μ using X and a presentation of the elementary diagram of L_β as a countable model.

NCR_1

Definition

- $L_{\omega_1^{CK}}$ denotes the collection of sets which are hyperarithmetically represented.
- $\mathcal{O}$ denotes the existential theory of $L_{\omega_1^{CK}}$—the complete Π^1_1-subset of ω.

Proposition (Well-known)

Every recursive closed game on ω^ω for which the closed player wins, has a winning strategy recursive in $\mathcal{O}$.

NCR_1 – Optimized Cone Argument

Lemma

The set

$$\{Y : \exists Z \exists R(R \text{ is 2-random in } Z \text{ and } Y \equiv_T Z \oplus R).\}$$

contains a closed subset of ω^ω whose degrees are cofinal in $\mathcal{D}$.

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contains a closed subset of ω^ω *whose degrees are cofinal in* $\mathcal{D}$.

Consequently, by the previous argument, if $Y \geq_T \mathcal{O}$,

$$\exists Z \exists R(R \text{ is 2-random in } Z \text{ and } Y \equiv_T Z \oplus R).\}$$

and so $Y \notin NCR_1$.

NCR_1 – Optimized Countability Argument

Theorem (Woodin)

If $X \in 2^\omega$ is not hyperarithmetic, then there is a $G \in 2^\omega$ such that $X \oplus G \geq \mathcal{O}^G$.

NCR_1 – Optimized Countability Argument

Theorem (Woodin)

If $X \in 2^\omega$ is not hyperarithmetic, then there is a $G \in 2^\omega$ such that $X \oplus G \geq \mathcal{O}^G$.

Theorem

If $X \in 2^\omega$ is not hyperarithmetic, then there is a representation m of a continuous measure μ such that X is 1-μ-random relative to m.