Randomness for Continuous Measures

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Motivation

Joint work with Jan Reimann.

Question

For which sequences $X \in 2^\omega$ *do there exist (representations of) continuous probability measures* μ *such that* X *is random for* μ*?*

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Example

Different measures give different notions of randomness. Student scores on a calculus exam do not produce a random sequence for the uniform probability distribution, but they do give a random sequence for an appropriately weighted distribution.

Randomness

There is a detailed and robust theory characterizing effective randomness relative to Lebesgue measure. Equivalently:

- *(Martin-Löf)* X is random if and only if X does not belong to any effectively presented set of measure 0.
- *(Kolmogorov et al)* X is random if and only if all of its initial segments are effectively incompressible.

Randomness

We will consider effective aspects of X's being random as applied to a variety of measures μ.

Definition

μ is *continuous* iff for every $X \in 2^\omega$, $\mu(\{X\}) = 0$.

The cases of discontinuous and continuous measures are quite different.

Notation

We will use finite binary sequences σ and subsets A of $2^{<\omega}$ to define open subsets of 2^{ω}:

$$[\sigma] = \{X : \sigma \text{ is an initial segment of } X\}$$

$$[A] = \{X : \exists\sigma \in A(X \in [\sigma])\}$$

Representing Measures

Let μ be a probability measure on 2^ω.

Definition

A *representation* m of a probability measure μ on 2^ω provides, for each $\sigma \in 2^{<\omega}$, a sequence of intervals with rational endpoints, each interval containing $\mu([\sigma])$, and with lengths converging monotonically to 0.

Effective μ-Randomness

Let m represent μ.

Definition

1. A *Martin-Löf test for μ relative to m* is a sequence $(A_n : n \geq 1)$ of subsets of $2^{<\omega}$ such that $(A_n : n \geq 1)$ is uniformly recursively enumerable relative to m and for each n, $\mu([A_n]) \leq 1/2^n$.
2. For $X \in 2^\omega$, X is *effectively μ-random relative to m* iff for every Martin-Löf test $(A_n : n \geq 1)$ for μ relative to m, $X \notin \cap_{n\geq 1}[A_n]$.

When m is understood, we will just speak of X's being μ random.

Higher Levels of Randomness

Definition

$X \in 2^\omega$ is n-random relative to a representation m of μ if and only if X passes every Martin-Löf test relative to $m^{(n-1)}$ (the $(n-1)$st Turing jump of m), in which the measures of the open sets of the test are evaluated using μ.

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We now have the basic definitions in place.

The Discontinuous Case

Now, we consider discontinuous probability measures. According to the previous definition, if μ concentrates on X then X is μ-random. However, this only occurs trivially.

Proposition

If $\mu(\{X\}) > 0$, then X is recursive relative to any representation of μ.

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Question

When is $X \in 2^\omega$ random relative to some μ for which X is not an atom?

General Measures

Theorem

For $X \in 2^\omega$, the following conditions are equivalent.

1. *There is a probability measure μ on 2^ω such that X is not a μ-atom and X is random relative to μ.*
2. *X is not recursive.*

General Measures

Positive. The only way to avoid being recursive is to have random content.

Negative. Relative recursive randomness can only distinguish between recursive and not recursive.

Fix X to be not recursive. In the next few frames, we will sketch the main ingredients of the construction of the measure μ from X.

Posner and Robinson

Theorem (Posner and Robinson)

For any nonrecursive X, there is a G such that $X + G \equiv_T G'$.

Fix G so that, relative to G, X has the same Turing degree as the Turing jump of G.

Kučera

Theorem (Kučera)

There is a 1-random set R such that $R \equiv_T 0'$.

Kučera's proof relativizes.

Relative to G, X has the same Turing degree as a random real R.

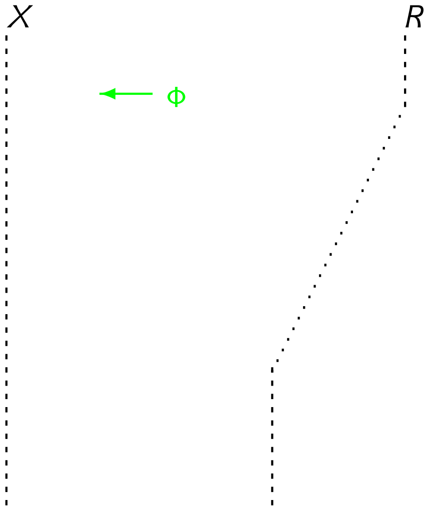
Pushing Randomness from R to X

Let Ψ and Φ be Turing functionals recursive relative to G such that $\Phi(R) = X$ and $\Psi(X) = R$.

If Φ were a homeomorphism with inverse Ψ, then there would be a measure μ obtained by pulling back to Lebesgue measure using $\Phi^{-1} = \Psi$. R's being random would ensure X's being μ-random.

We adapt this paradigm to the partial continuous Φ and Ψ, which are inverses on X and R.

Pushing Randomness from R to X



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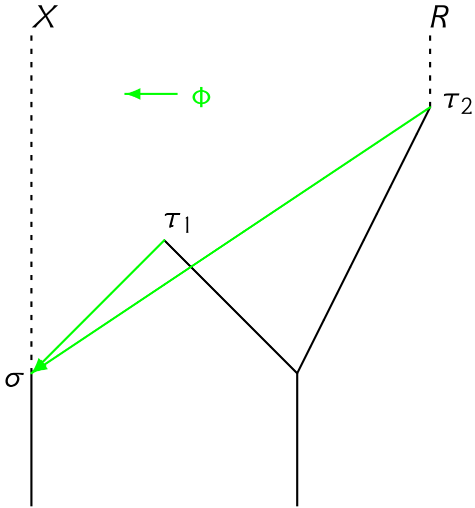

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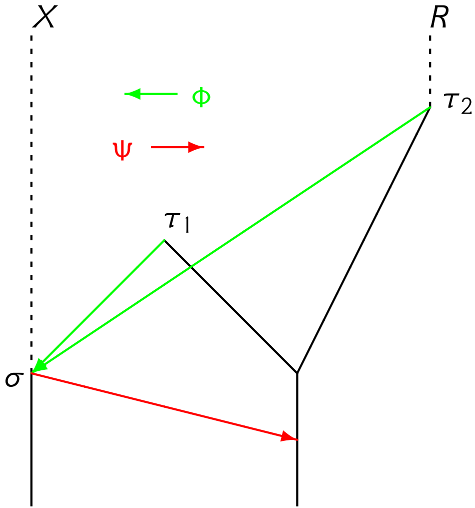

Pushing Randomness from R to X

For $\sigma \in 2^{<\omega}$, let $Pre(\sigma)$ be the set of minimal elements of

$$\{\tau : \Phi(\tau) = \sigma \text{ and } \Psi(\sigma) \subseteq \tau\}.$$

When X extends σ, $Pre(\sigma)$ is a recursively enumerable antichain of possible initial segments of R.
In the previous slide, τ_1 and τ_2 are elements of $Pre(\sigma)$.

Pushing Randomness from R to X

Let λ denote Lebesgue measure. Consider the following consistency requirements $\mathcal{R}$ on a measure μ.

1. $\mu([\sigma]) \geq \lambda([Pre(\sigma)])$. Thus, μ dominates the measure of pulling back Φ on those strings for which $\Psi(\Phi)$ is the identity.
2. $\mu([\sigma]) \leq \lambda([\Psi(\sigma)])$. Thus, μ does not concentrate on reals in the domain of Ψ.

Pushing Randomness from R to X

There is an infinite G-recursive, G-recursively-bounded tree T such that any infinite path in T is a rational representation m of a measure μ satisfying $\mathcal{R}$.

Lemma (Downey-Hirschfeldt-Miller-Nies, Reimann-Slaman)

Any infinite G-recursive, G-recursively-bounded tree has an infinite path m such that R is random relative to m.

Pushing Randomness from R to X

Fix a path m in T such that R is random relative to m. X's failing an m-recursive Martin-Löf test relative to μ would pull back to R's failing an m-recursive Martin-Löf test relative to λ, an impossibility.

Conclusion: X is μ-random.

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Definition

- For X, Y, and Z in 2^ω, we write $X \equiv_{T,Z} Y$ to indicate that there are Turing reductions Φ and Ψ which are recursive in Z such that $\Phi(X) = Y$ and $\Psi(Y) = X$.
- When Φ and Ψ are total, we write $X \equiv_{tt,Z} Y$.

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- When Φ and Ψ are total, we write $X \equiv_{tt,Z} Y$.

Turing reductions correspond to continuous functions defined on subsets of 2^ω. Truth-table (tt) reductions correspond to continuous functions defined on all of 2^ω.

Continuous Measures

degree theoretically characterizing relative randomness

Proposition

For X and Z in 2^ω, the following conditions are equivalent.

- *There is a continuous measure μ which is recursive in Z such that X is n-random for μ and Z.*

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- *There is a continuous dyadic measure μ which is recursive in Z such that X is n-random for μ and Z.*
- *There is an R such that R is n-random relative to Z and an order preserving homeomorphism $f : 2^\omega \to 2^\omega$ such that f is recursive in Z and $f(R) = X$.*

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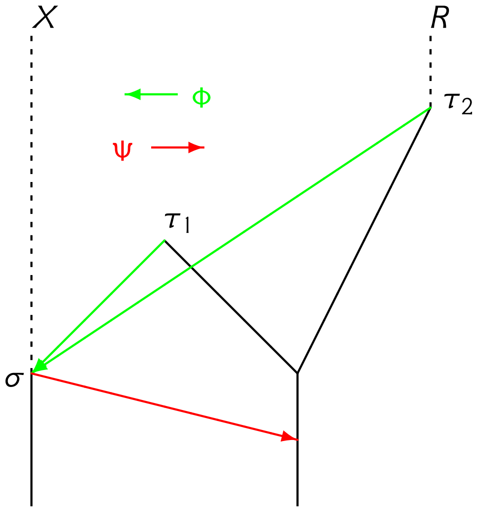
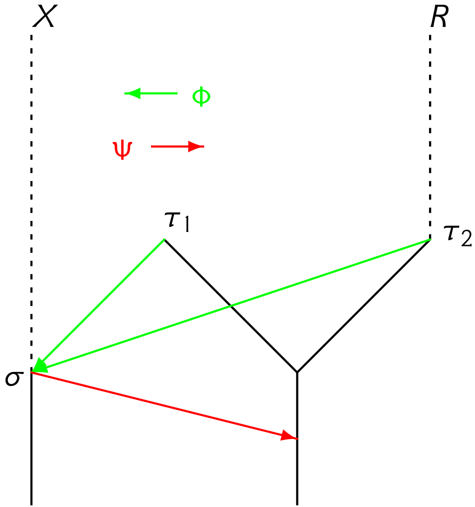
- *There is an R such that R is n-random relative to Z and an order preserving homeomorphism $f : 2^\omega \to 2^\omega$ such that f is recursive in Z and $f(R) = X$.*
- *There is an R such that R is n-random relative to Z and $X \equiv_{tt,Z} R$.*

$X \equiv_T R$



When Ψ and Φ are partial recursive, $Pre(\sigma)$ is a recursively enumerable set.

$X \equiv_{tt} R$



When Ψ and Φ are total recursive, $Pre(\sigma)$ is a recursive set as is the set $\{\tau : \Psi(\Phi(\tau)) \not\subseteq \tau\}$. The former contributes to $\mu(\sigma)$ and the latter can be distributed as it appears. Since Ψ is total, μ is continuous.

Recursive Closed Sets

Definition

For T is a recursive subtree of $2^{<\omega}$, then the set P of infinite paths through T is a Π^0_1-class.

Theorem (Well-known)

Suppose that R is 1-random, P is a Π^0_1-class, and $R \in P$. Then $\lambda(P) > 0$.

Failures of Continuous Randomness

Theorem (Kjos-Hanssen and Montalbán)

Suppose that P is a countable Π^0_1-class and $X \in P$. Then there is no continuous μ such that X is 1-μ-random.

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Suppose that P is a countable Π^0_1-class and $X \in P$. Then there is no continuous μ such that X is 1-μ-random.

Definition

$X \in NCR_k$ if and only if there is no representation m of a continuous measure μ such that X is k-random relative to the representation m of μ.

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$X \in NCR_k$ if and only if there is no representation m of a continuous measure μ such that X is k-random relative to the representation m of μ.

By Kjos-Hanssen and Montalbán, every element of a countable Π^0_1-class belongs to NCR_1.