On the reals never continuous random

Liang Yu

Institute of Mathematical Sciences
Nanjing University

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Probability measure

Definition

A measure ρ over 2^ω is *probability measure* if

1. $\rho(\emptyset) = 1$; and
2. For any $\sigma \in 2^{<\omega}$, $\rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$.

Continuous measure

Definition

Given a measure ρ, a real x is an *atomic* respect to ρ if $\rho(\{x\}) > 0$.

Definition

A measure ρ is *continuous* if $\forall x \rho(\{x\}) = 0$.

Representing measures

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A measure ρ is recursive if its representation is recursive.

Randomness under general probability measures

For a fixed probability measure ρ, we may define Martin-Löf randomness respect to ρ.

NCR

Definition

A real x is never continuous random (or *NCR*), if x cannot be random respect to any continuous measure.

NCR is countable

Theorem (Reimann and Slaman)

Every real in NCR is hyperarithmetic.

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The proof is based on Woodin's result that every nonhyperarithmetic real *tt*-cups a real z to $\mathcal{O}^z$, the hyperjump relative to z.

Higher randomness

Definition

1. A real x is Δ^1_1-random if it does not belong to any Δ^1_1-null set.
2. A real x is Π^1_1-random if it does not belong to any Π^1_1-null set.

Some basic facts

Theorem (Sacks)

$\{x \mid x \geq_h \mathcal{O}\}$ is a Π^1_1 null set.

Theorem (Kechris; Stern; Hjorth and Nies)

There is a largest Π^1_1-null set.

Theorem (Stern; Chong, Nies and Yu)

A real x is Π^1_1-random iff x is Δ^1_1-random and $\omega_1^x = \omega_1^{\mathrm{CK}}$ iff x is Δ^1_1-random and every function Δ^1_1 in x is dominated by a Δ^1_1-function.

On $NCR_{\Pi^1_1}$

Theorem (Chong and Yu)

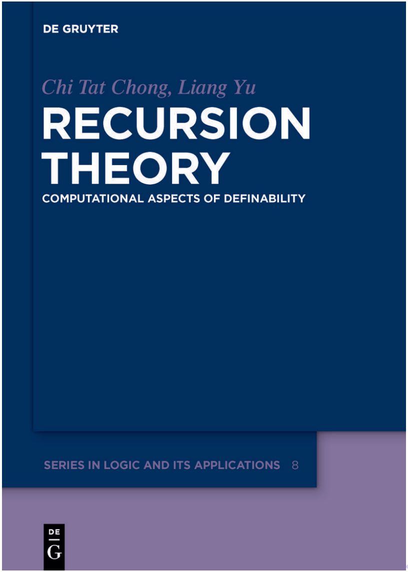
$NCR_{\Pi^1_1} = \{x \mid x \in L_{\omega_1^x}\}$.

Proof.

- $NCR_{\Pi^1_1}$ is a Π^1_1-set.
- $NCR_{\Pi^1_1}$ does not contain a perfect subset.
- If $x \in L_{\omega_1^x}$, then for any continuous measure ρ, either $\rho \geq_h x$ or $x \oplus \rho \geq_h \mathcal{O}^\rho$.

□

To fully understand these facts



L-randomness

Definition

A real x is L-random if for any L-coded sequence open sets $\{U_n\}_{n\in\omega}$ with $\forall n\mu(U_n) < 2^{-n}$, $x \notin \bigcap_n U_n$.

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So an L-random real is exactly a Solovay real.

NCR_L

Proposition (Yu and Zhu)

- *NCR_L is a Π^1_3-set.*
- *If $NCR_L \neq 2^\omega$, then it is not Π^1_2.*
- *If x is L-random and $y \in L[x] \setminus L$, then $y \notin NCR_L$.*
- *If $V = L[r]$ for some L-random real r, then NCR_L is a proper Σ^1_2-set.*
- *If for any real x, $(2^\omega)^{L[x]}$ is countable, then*
    - *NCR_L does not contain a perfect subset.*
    - *NCR_L is Σ^1_2 if and only if $NCR_L \subseteq L$.*

The third item follows from a set theoretic version of Demuth Theorem.

Under *PD*

Proposition

Every Π^1_2-singleton belongs to NCR_L. Actually if A is a countable Π^1_2-set, then $A \subseteq NCR_L$.

Proof.

By Shoenfield absoluteness. □

Some examples

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Theorem (Friedman)

There is a nonconstructible Π^1_2*-singleton* $x <_L 0^\sharp$.

By Friedman, there exists a non-Π^1_2-singleton belonging to a countable Π^1_2-set.

L-jumps

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Definition

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Note that $x \in P_2 \implies x^\sharp \in P_2$.
$0, 0^\sharp, (0^\sharp)^\sharp, \cdots \in P_2$.

$P_2 \subseteq NCR_L$.

Proposition (Yu and Zhu)

$P_2 \subseteq NCR_L$.

Proof.

If x is L-random respect to ρ, then $\kappa^x \leq \kappa^\rho$. So $x \in L[\rho]$, a contradiction. □

Q_3

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By the previous result, $Q_3 \not\subseteq NCR_L$.

$NCR_L \subseteq Q_3$ (I)

Theorem (Yu and Zhu)

$NCR_L \subseteq Q_3$.

Lemma

For any real x, there is a real $y \geq_T x$ so that there is a continuous measure $\rho \leq_T y$ so that y is L-random respect to ρ.

Let

$$B = \{y \mid \exists \rho \leq_T y(\rho \text{ is continuous and } y \text{ is } L\text{-random respect to } \rho)\}.$$

Then B is a Π^1_2 set and has cofinally many L-degrees.

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B is co-uncountable.

$NCR_L \subseteq Q_3$ (II)

Let $D = \{y_0 \mid \forall y(y \geq_T y_0 \rightarrow y \in B)\}$ be a nonempty Π^1_2-set.
B contains the Q_3-complete real $y_{0,3}$ which is a base for D.

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The relativized version is read as that for any real z, the set $B_z = \{y \mid \exists \rho \leq_T y \oplus z(\rho \text{ is continuous and } y \text{ is } L[z]\text{-random respect to } \rho)\}$ contains an upper cone with the base $y_{z,3}$.

$NCR_L \subseteq Q_3$ (III)

A higher version of Posner-Robinson Theorem.

Theorem (Woodin)

If $x \notin Q_3$, then there is a real z so that $x \oplus z \geq_{tt} y_{z,3}$.

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If $x \notin Q_3$, then there is a real z so that $x \oplus z \geq_{tt} y_{z,3}$.

Then for any real $x \notin Q_3$, there is a real z so that $x \oplus z$ is $L[z]$-random respect to some continuous measure $\rho \leq_T x \oplus z$.

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Applying Demuth's technique, we have that x is $L[z]$-random respect to some continuous measure $\rho_0 \leq_L z \oplus \rho$.

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Applying Demuth's technique, we have that x is $L[z]$-random respect to some continuous measure $\rho_0 \leq_L z \oplus \rho$.

So $NCR_L \subseteq Q_3$.

M_1

Theorem (Steel)

1. *There exists the least inner model, denoted by M_1, containing a Woodin cardinal.*
2. *The reals in M_1 are precisely Q_3.*

$\tilde{P}_2$

Definition

$\tilde{P}_2 = \{x \mid \exists y(x \equiv_L y \wedge y \text{ is a master code in } \mathrm{M}_1)\}$.

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$\tilde{P}_2 = \{x \mid \exists y(x \equiv_L y \wedge y \text{ is a master code in } \mathrm{M}_1)\}$.

Obviously $\tilde{P}_2$ has cofinally many L-degrees in M_1.

Proposition (Y and Zhu)

$\tilde{P}_2 \subseteq P_2$.

So NCR_L has cofinally many L-degrees in Q_3 and so is properly Π^1_3.

A conjecture

Conjecture

$\tilde{P}_2 = P_2$?

Thanks