Irrationality Exponents and Effective Hausdorff Dimension

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Celebrating Rod Downey's Mathematical Contributions

Ralph Waldo Emerson on the purpose of life:

> *It is to be useful, to be honorable, to be compassionate, to have it make some difference that you have lived and lived well.*

Cheers Rod on an exemplary mathematical life:

- ⋆ Practitioner
- ⋆ Expositor
- ⋆ Mentor
- ⋆ Leader

Abstract

Suppose $a \geq 2$ and $b \in [0, 2/a]$.

- (Generalization of Jarník 1929 and Besicovitch 1934) There is a Cantor-like set with Hausdorff dimension equal to b such that, with respect to its uniform measure, almost all real numbers have irrationality exponent equal to a.
- There is a Cantor-like set such that, with respect to its uniform measure, almost all real numbers have effective Hausdorff dimension equal to b and irrationality exponent equal to a.

In each case, we obtain the desired set as a distinguished path in a tree of Cantor sets.

Hausdorff Dimension

For a set of real numbers X and a non-negative real number s the *s-dimensional Hausdorff measure of X* is defined by

$$\liminf_{\varepsilon \to 0} \left\{ \sum_{j \geq 1} r_j^s : \begin{array}{c} \text{there is a cover of } X \text{ by balls with} \\ \text{radii } (r_j : j \geq 1) \text{ and } \forall j (r_j < \varepsilon) \end{array} \right\}.$$

The *Hausdorff dimension* of X is the infimum of the set of non-negative reals s such that the s-dimensional Hausdorff measure of X is zero.

Effective Hausdorff Dimension of $\xi \in 2^{\mathbb{N}}$

Definition

The *effective Hausdorff dimension* of a real number ξ is the infimum of the set of t such that there is a c for which there are infinitely many ℓ such that the prefix-free Kolmogorov complexity of the first ℓ digits in the binary expansion of ξ is less than $t \cdot \ell + c$.

Heuristic: The effective Hausdorff dimension of a real number ξ is the infimum of the algorithmic compression factors of the initial segments of the binary expansion of ξ.

- Computable real numbers have effective dimension 0.
- Random real numbers have effective dimension 1.
- The set of real numbers with effective Hausdorff dimension b has Hausdorff dimension b.

There is an equivalent formulation using effectively presented covers.

Irrationality Exponent

Definition (originating with Liouville 1855)

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers a such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^a}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

- When a is large and $0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^a}$, then p/q is a good approximation to ξ when considered in the scale of $1/q$.
- The irrationality exponent of ξ is a indicator for how well ξ can be approximated by rational numbers (a linear version of Kolmogorov complexity).

Examples

- Random real numbers have irrationality exponent equal to 2.
- (Roth 1955) Irrational algebraic real numbers have irrationality exponent equal to 2.
- Liouville numbers are those with infinite irrationality exponent—these were the first examples of transcendental numbers.

Example

For $a \geq 2$, $\{\xi : \xi$ has irrationality exponent $a\}$ has Hausdorff dimension less than or equal to $2/a$.

Consequences of Irrationality Exponent for Effective Dimension

Remark

If ξ has irrationality exponent equal to a, then ξ has effective Hausdorff dimension less than or equal to $2/a$:

Proof

- Say that $|p/q - \xi| < 1/q^a$.
- Need $2 \cdot \log_2 q$ bits to specify p and q.
- Obtain $a \cdot \log_2 q$ bits in the binary expansion of ξ.
- $\dfrac{2 \cdot \log_2 q}{a \cdot \log_2 q} = 2/a$.

No Other Consequences

The second result mentioned earlier has the following corollary.

Theorem (Becher, Reimann and Slaman)

For every $a \geq 2$ and every b in $[0, 2/a]$, there is a real number ξ such that ξ has irrationality exponent a and effective Hausdorff dimension b.

The Jarník-Besicovitch Theorem

Theorem (Jarník 1929 and Besicovitch 1934)

For every real number a greater than or equal to 2, the set of numbers with irrationality exponent equal to a has Hausdorff dimension exactly equal to '$2/a$.

As mentioned earlier, it is a direct application of the definitions to show that the Hausdorff dimension of the set of numbers with irrationality exponent a is less than or equal to $2/a$. The other inequality comes from an early application of fractal geometry.

Jarník's Fractal

For each real number a greater than 2, Jarník gave a Cantor-like construction of a fractal J contained in $[0,1]$ of Hausdorff dimension $2/a$ such that the uniform measure ν on J satisfies the following:

- Every element of J has irrationality exponent greater than or equal to a.
- For all b greater than a, the set of numbers with irrationality exponent greater than or equal to b has ν-measure equal to 0.

Let $(M_i : i \in \mathbb{N})$ be a rapidly increasing sequence of natural numbers.

Define $(E_i : i \in \mathbb{N})$ as follows.

- $E_0 = [0,1]$
- For $i > 0$, let

$$E_i = \bigcup \left\{ [p/q - 1/q^a, p/q + 1/q^a] : \begin{array}{c} 0 < p < q, M_i \leq q \leq 2M_i, q \text{ prime}, \\ [p/q - 1/q^a, p/q + 1/q^a] \subset E_{i-1} \end{array} \right\}$$

Let $J = \bigcap_{i \in \mathbb{N}} E_i$.

The Mass Distribution Principle

Every element of J has irrationality exponent less than or equal to a, so the Hausdorff dimension of J is less than or equal to $2/a$.

Show that J has Hausdorff dimension at least $2/a$ by applying the following fact for the uniform measure μ on J.

Theorem (Mass Distribution Principle)

Let ν be a finite measure, d a positive real number and X a set with Hausdorff dimension less than d. Suppose that there is a positive real number C such that for every interval I, $\nu(I) < C\,|I|^d$. Then $\nu(X) = 0$.

Modifying J – Version 1

For $a \geq 2$ and $b \in [0, 2/a]$, there is a Cantor-like set with Hausdorff dimension equal to b such that, with respect to its uniform measure, almost all real numbers have irrationality exponent equal to a.

Find $J_1 \subset J$ by thinning the levels of J, either by using fewer primes or by using one prime and fewer intervals $[p/q - 1/q^a, p/q + 1/q^a]$ and let μ_1 be the uniform measure on J_1.

- Ensure that the intervals from E_i which are retained to form J_1 provide the covers needed to show that J_1 has Hausdorff dimension less than or equal to b.
- Ensure the MDP for μ_1 with exponent b , and thereby ensure that J_1 has Hausdorff dimension exactly equal to b.
- Ensure that μ_1-almost all elements of J_1 have irrationality exponent equal to a by choosing from among all possible thinnings the one that minimizes the frequency of occurrences of rational approximation with exponent greater than a.

Modifying J – Version 2

For $a \geq 2$ and $b \in [0, 2/a]$, there is a Cantor-like set such that, with respect to its uniform measure, almost all elements in the set have effective Hausdorff dimension equal to b and irrationality exponent equal to a.

Find $J_2 \subset J$ by thinning the levels of J, either by using fewer primes or by using one prime and fewer intervals $[p/q - 1/q^a, p/q + 1/q^a]$ and let μ_2 be the uniform measure on J_1.

- Stratify the construction of J_2 into extended computable blocks of dimension close to b, thereby producing for each element of J_2 instances of algorithmic compression approaching b and ensuring that μ_2-almost every element of J_2 has effective Hausdorff dimension less than or equal to b.
- Ensure the MDP for μ_2 with exponent b. Thus, for $d < b$, the set of real numbers with effective Hausdorff dimension equal to d is a μ_2-null set and so μ_2-almost every element of J_2 has effective Hausdorff dimension exactly b.
- Ensure that μ_2-almost all elements of J_2 have irrationality exponent equal to a as before.

The End