Conservativity of ultrafilters over subsystems of second order arithmetic

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There are now many uses of ultrafilters in proving combinatorial results whose statements lie well within second order arithmetic, Z_2. In the spirits of classical combinatorics and reverse mathematics, we want to find "elementary" proofs and classify the theorems in terms of the subsystems of Z_2 needed to prove them.

One approach (Hirst and others) was to replace the ufs in the classical proofs with countable approximations inside models of some subsystem of Z_2. Following Towsner we analyze a more wholesale approach of proving conservation results over subsystems of Z_2 for corresponding axiom systems with ufs.

The Settings

We want a language $\mathcal{L}^{\mathcal{U}}$ extending the usual one $\mathcal{L}$ for Z_2 in which we can talk about ultrafilters. For T a typical subsystem of Z_2, we want a natural extension $T^{\mathcal{U}}$ to the language $L^{\mathcal{U}}$ asserting the existence of some type of ultrafilter. Then we want to prove that $T^{\mathcal{U}}$ is Γ-conservative over T for natural subclasses Γ of the sentences of $\mathcal{L}$. Thus, if we can prove the desired combinatorial theorem using an ultrafilter as described by $T^{\mathcal{U}}$, we can prove it just in T.

Towsner carried out such an program for some kinds of ultrafilters and $T = ACA_0, ATR_0$ and $\Pi^1_1 - CA_0$. His approach used a syntactic and proof-theoretic view of forcing. (He also mentions related results using model theoretic methods by Enayat and functional interpretation by Kreuzer.) Ours view is a semantic one that at times gives additional results and information. We also consider additional types of ultrafilters. While writing this up we found that Ramsey ufs had made it into the printed version of Towsner's paper and that Kreuzer had extended the proof-theoretic approach to even more ufs within other axiomatic systems.

Some Combinatorial Notions

We skip the usual listing of the subsystems $RCA_0, ACA_0, ACA_0^+, ATR_0$ and $\Pi^1_n - CA_0$ of Z_2 but give some basic notions and results from combinatorics. We begin with Hindman's theorem, variants and generalizations.

Let $\mathcal{M} = \langle M, S, +, \times, <, 0, 1, \in\rangle$ be a structure for $\mathcal{L}$ with $\mathcal{M} \vDash RCA_0$.

Hindman's Theorem (HT): If f is a finite coloring of M, i.e. $f : M \to k$, then there is an infinite *homogeneous* $X \subseteq M$ s.t. $FS(X)$, the set of sums of finite subsets of X is monochromatic, i.e. there is an $i < k$ such that for every finite sum z of distinct elements of X, $f(z) = i$.

Finite Union Theorem (FU): Same but for $[M]^{<\omega}$ and the union operations replacing M and $+$.

Galvin-Glazer (GG): Same but for fairly arbitrary semigroup $\langle Z, *\rangle$ in place of $\langle M, +\rangle$.

Iterated Hindman's Theorem (IHT): Given $\langle f_n, k_n \rangle$ with f_n a k_n coloring of M, there is a *homogeneous* infinite subset $X = \langle x_i \rangle$ of M such that, for each n, $FS(\{x_i | i \geq n\})$ is monochromatic for f_n.

Iterated Finite Union (IFU) and Iterated Galvin-Glazer (IGG) are defined analogously.

Theorem ($\mathbf{RCA_0}$): HT, FU and GG are equivalent; so too are IHT,IFU and IGG.

Definition: An *$\mathcal{M}$-filter* is a nonempty subset $\mathcal{F}$ of S which is closed upward and under intersection, i.e. if $X, Y \in S$, $X \subseteq Y$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$; and if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$.

An $\mathcal{M}$-filter is *nonprincipal* if there is no $A \in S$ such that $\mathcal{F} = \{X \in S | X \supseteq A\}$.

A nonprincipal $\mathcal{M}$-filter $\mathcal{U}$ is an *$\mathcal{M}$-ultrafilter* if $\forall A \in S(A \in \mathcal{U}$ or $A^c \in \mathcal{U})$.

An $\mathcal{M}$-ultrafilter $\mathcal{U}$ is *Ramsey* (*selective)* if for every partition $\langle X_n \rangle \in S$ of M into nonempty pairwise disjoint sets such that $X_n \notin \mathcal{U}$ for every n, there is a $Z \in \mathcal{U}$ such that $|X_n \cap Z| = 1$ for every n.

An $\mathcal{M}$-ultrafilter $\mathcal{U}$ is *idempotent* **(wrt** $+$) if $\forall X \in \mathcal{U}$, $\{n | (X - n) \in \mathcal{U}\} \in \mathcal{U}$ where $(X - n) = \{m | m + n \in X\}$.

Syntax, Semantics and Theories

Language: Add to $\mathcal{L}$ a unary function symbol $\delta_{\mathcal{U}}$ from sets to sets. So we have new terms $\delta_{\mathcal{U}}(X)$ (and $\delta_{\mathcal{U}}^{n}(X)$ for n-fold iterations with $n \in \mathbb{N}$) and associated new atomic formulas.

Structures: Begin with a structure $\mathcal{M} = \langle M, S, +, \times, <, 0, 1, \in \rangle$ for $\mathcal{L}$. Fix a $\mathcal{U} \subseteq S$ and interpret $\delta_{\mathcal{U}}(X)$ as the subset of $\mathcal{M}$ given by $0 \in \delta_{\mathcal{U}}(X) \Leftrightarrow X \in \mathcal{U}$ and $n+1 \in M \Leftrightarrow X^{[n]} \in \mathcal{U}$. So we are coding $X \in \mathcal{U}$ and $X^{[n]} \in \mathcal{U}$ into $\delta_{\mathcal{U}}(X)$. We require that S is closed under this operation to get a structure for $\mathcal{L}^{\mathcal{U}}$.

Theories: If T is a typical subsystem of Z_2, then $(T^{I\mathcal{U}})T^{\mathcal{U}}$ has all the axioms of T but we also allow formulas of $\mathcal{L}^{\mathcal{U}}$ from the same syntactic class as those of $\mathcal{L}$ included in the comprehension axioms of T. In addition, $(T^{I\mathcal{U}})T^{\mathcal{U}}$ has axioms saying $\mathcal{U}$ is an (idempotent) ultrafilter $(\mathcal{AU}1 - \mathcal{AU}5)$ $\mathcal{AU}1 - \mathcal{AU}4$:

Axioms for Ultrafilters

$\mathcal{AU}1$ $\forall X(0 \in \delta_{\mathcal{U}}(X) \rightarrow \forall x \exists y(y > x \ \wedge \ y \in X)$. (Every set in $\mathcal{U}$ is infinite.)

$\mathcal{AU}2$ $\forall X \forall Y(X \subseteq Y \ \wedge \ 0 \in \delta_{\mathcal{U}}(X) \rightarrow 0 \in \delta_{\mathcal{U}}(Y))$. ($\mathcal{U}$ is closed under supersets.)

$\mathcal{AU}3$ $\forall X \forall Y \forall Z(0 \in \delta_{\mathcal{U}}(X) \ \wedge \ 0 \in Y \ \wedge \ X \cap Y = Z \rightarrow 0 \in \delta_{\mathcal{U}}(Z))$. ($\mathcal{U}$ is closed under intersections.)

$\mathcal{AU}4$ $\forall X \forall Y(Y = \bar{X} \rightarrow (0 \in \delta_{\mathcal{U}}(X) \ \vee \ 0 \in \delta_{\mathcal{U}}(Y)$. (For every X, X or $\bar{X}$ is in $\mathcal{U}$.)

The next axiom guarantees that $\mathcal{U}$ is idempotent.

$\mathcal{AU}5$ $\forall X \forall Y \forall Z(0 \in \delta_{\mathcal{U}}(X) \ \wedge \ \forall n(Y^{[n]} = X - n)) \ \wedge \ \forall i(i \in Z \Leftrightarrow i + 1 \in \delta_{\mathcal{U}}(Y)) \rightarrow 0 \in \delta_{\mathcal{U}}(Z))$. (If $X \in \mathcal{U}$ then $\{n | (X - n) \in \mathcal{U}\} \in \mathcal{U}$.)

A sample but motivating result:

Theorem: $ACA_0^{I\mathcal{U}} \vdash IHT$. The proof mimics that of Galvin and Glazer.

Towards Conservation Results

As we show that $\mathsf{RCA}_0^{\mathcal{U}} \vdash \mathsf{ACA}_0^{\mathcal{U}}$ and our goal is to prove conservation results for $T^{I\mathcal{U}}$ over T, the weakest theory T worth considering is ACA_0. So we assume that all structures $\mathcal{M}$ are models of ACA_0.

We prove more than a simple conservation results for most T of interest: Every (countable) model $\mathcal{M}$ of T can be extended to one of $T^{I\mathcal{U}}$ simply by adding on a subset $\mathcal{U}$ of S which is an idempotent $\mathcal{M}$-ultrafilter and interpreting $\delta_{\mathcal{U}}(X)$ so that $0 \in \delta_{\mathcal{U}}(X) \Leftrightarrow X \in \mathcal{U}$ and $n+1 \in M \Leftrightarrow X^{[n]} \in \mathcal{U}$.

This immediately gives that $T^{I\mathcal{U}}$ is conservative over T for all sentences of Z_2.

Our plan is to construct $\mathcal{U}$ by a forcing argument over a model $\mathcal{M}$ of T.

Notions of Forcing

Definition: The conditions of $\mathbb{P}_{I\mathcal{U}}$ are sequences $u = \langle U_0, U_1, \ldots \rangle$ in $\mathcal{M}$ such that there are $y_0 < y_1 < y_2 < \ldots$ such that, for every i, $U_i = FS(y_i, y_{i+1}, \ldots)$. We define extension for $\mathbb{P}_{I\mathcal{U}}$ by $v = \langle V_i \rangle \leq \langle U_i \rangle = u \Leftrightarrow \forall i \exists j (V_i \supseteq U_j)$.

Both membership and extension in $\mathbb{P}_{I\mathcal{U}}$ are arithmetic relations in $\mathcal{M}$.

We are thinking of u as representing the filter $F_u = \{A \in S | \exists i (A \supseteq U_i)\}$ It is easy to see that this set is a nonprincipal $\mathcal{M}$-filter for any $u \in \mathbb{P}_{I\mathcal{U}}$ and, for example, if $v \leq u$ then $F_v \supseteq F_u$.

We write $A \in u$ to mean that $\exists i (A \supseteq U_i)$, i.e. $A \in F_u$. Given any ($\mathcal{M}$-generic) filter $\mathbb{U}$ on $\mathbb{P}_{I\mathcal{U}}$, the corresponding ($\mathcal{M}$-generic) object $\mathcal{U}$ is $\{A \in S | (\exists u \in \mathbb{U})(A \in u)\}$.
As our goal is to extend $\mathcal{M}$ to a model of $T^{I\mathcal{U}}$ without changing the sets S of $\mathcal{M}$, it is natural to use the countable chain condition for $\mathbb{P}_{I\mathcal{U}}$.

Lemma: If u_i is a descending sequence in $\mathbb{P}_{I\mathcal{U}}$ then there is a condition u extending every u_i. Proof: direct from the definitions.

Defining the Forcing Relation

We begin with the idea of deciding a set X.

Definition: A condition $u \in \mathbb{P}_{I\mathcal{U}}$ *decides a set* $X \in S$ if $(X \in u \vee \bar{X} \in u)$ and $\forall n(X^{[n]} \in u \vee \overline{X^{[n]}} \in u)$.

Proposition: If u decides X, then there is a $Y \in S$ such that $0 \in Y \Leftrightarrow X \in u$ and $n+1 \in Y \Leftrightarrow X^{[n]} \in u$. We then say that $u \Vdash \delta_{\mathcal{U}}(X) = Y$. The relations u decides X, Y is the set such that $u \Vdash \delta_{\mathcal{U}}(X) = Y$ and $u \Vdash \delta_{\mathcal{U}}(X) = Y$ are uniformly arithmetic (in u, X and Y as relevant). Proof: Basically follows from the definitions.

Notation: Let $\delta^0_{\mathcal{U}}(X) = X$ and $\delta^{n+1}_{\mathcal{U}}(X) = \delta_{\mathcal{U}}(\delta^n_{\mathcal{U}}(X))$. By we say u *decides* $\delta^{n+1}_{\mathcal{U}}(X)$ if u decides Y for the Y such that $u \Vdash \delta^n_{\mathcal{U}}(X) = Y$ where we are also defining $u \Vdash \delta^n_{\mathcal{U}}(X) = Y$ by the obvious induction. These relations are also uniformly arithmetic by induction.

Defining the Forcing Relation

We start the definition of the forcing relation with the arithmetic formulas. **Definition:** Let $\Phi(Z_1, \ldots Z_m)$ be an arithmetic formula of $\mathcal{L}^{\mathcal{U}}$ with the free second order variables displayed. Consider an instance of $\Phi(X_1, \ldots X_m)$ specified by choosing values $X_i \in S$ for the Z_i and a condition u. We say that $u \Vdash \Phi(X_1, \ldots X_m)$ if u decides every $\delta_{\mathcal{U}}^{n_j}(X_i)$ occurring in Φ and $\mathcal{M}$ satisfies the sentence gotten from $\Phi(X_1, \ldots X_m)$ by substituting the formula saying that $k \in Y_{i,j}$ for the $Y_{i,j}$ such that $u \Vdash \delta_{\mathcal{U}}^{n_j}(X_i) = Y_{i,j}$ for the new atomic formulas $k \in \delta_{\mathcal{U}}^{n_j}(X_i)$ and $Y_{i,j} = Y_{k,l}$ for $\delta_{\mathcal{U}}^{n_j}(X_i) = \delta_{\mathcal{U}}^{n_k}(X_l)$.

We now proceed by induction to define forcing for second order sentences of $\mathcal{L}^{\mathcal{U}}$ (which we assume to be in prenex normal form) in the obvious way.

Definition: We say $u \Vdash \exists Z \Phi((X_1, \ldots X_m, Z)$ if there is an $X_{m+1} \in S$ such that $u \Vdash \Phi((X_1, \ldots X_m, X_{m+1})$. On the $\forall$ side we say $u \Vdash \forall Z \Phi((X_1, \ldots X_m, Z)$ if there is no $v \leq u$ and no $X_{m+1} \in S$ such that $v \Vdash \Phi((X_1, \ldots X_m, X_{m+1})$.

Complexity, Density and Models

Proposition: For arithmetic sentences $\Phi(X_1, \ldots X_m)$ of $\mathcal{L}^{\mathcal{U}}$, the relation $u \Vdash \Phi(X_1, \ldots X_m)$ is uniformly arithmetic in u and $X_1, \ldots X_m$. For Σ^1_n (Π^1_n) sentences $\Phi(X_1, \ldots X_m)$ of $\mathcal{L}^{\mathcal{U}}$ the relation $u \Vdash \Phi(X_1, \ldots X_m)$ is uniformly Σ^1_n (Π^1_n). (By the definitions.)

Proposition: ($\mathcal{M} \vDash IHT$) For each sequence $X_m \in S$, the set $\{u|\forall m(u$ decides $X_m)\}$ is dense in $\mathbb{P}_{I\mathcal{U}}$. For each sentence Φ of $\mathcal{L}^{\mathcal{U}}$ (with set parameters), the set $\{u|u$ decides $\Phi\}$ is also dense in $\mathbb{P}_{I\mathcal{U}}$.

Proof: Use some manipulations with addition and then apply IGG.

Theorem. If $\mathcal{M} \vDash IHT$ and $\mathbb{U}$ and the associated $\mathcal{U}$ are $\mathcal{M}$-generic for $\mathbb{P}_{I\mathcal{U}}$ then $\mathcal{M}^{\mathcal{U}}$ is an $\mathcal{L}^{\mathcal{U}}$ structure and a model of $\mathrm{ACA}_0^{I\mathcal{U}}$. Moreover, any sentence Φ of $\mathcal{L}^{\mathcal{U}}$ (with parameters) is true in $\mathcal{M}^{\mathcal{U}}$ if and only if there is a $u \in \mathbb{U}$ such that $u \Vdash \Phi$. Finally, $\mathcal{U}$ is an idempotent $\mathcal{M}$-ultrafilter.

Proof: The interesting part is taking a new comprehension axiom for $X = \{n|\Phi(n)\}$ and noting that deciding all instances is dense. Once decided, the truth of the instances is uniformly arithmetic in the condition u and so the desired X is arithmetic in u and exists in $\mathcal{M}$ by ACA_0.

Conservation Results

Corollary: $\mathrm{ACA}_0^{I\mathcal{U}}$ is a conservative extension of IHT for all sentences of second order arithmetic.

Proof: We already know that $ACA_0^{I\mathcal{U}} \vdash IHT$. So suppose $ACA_0^{I\mathcal{U}} \vdash \Phi$. The Theorem shows that every model of IHT (which implies ACA_0) can be extended to one $\mathcal{M}^{\mathcal{U}}$ of $\mathrm{ACA}_0^{I\mathcal{U}}$ with the same M and S. So if for any sentence Φ of second order logic there is an $\mathcal{M} \vDash IHT \ \wedge \ \neg\Phi$ then for any $\mathcal{M}$-generic $\mathcal{U}$, $\mathcal{M}^{\mathcal{U}} \vDash \neg\Phi \wedge \mathrm{ACA}_0^{I\mathcal{U}}$ for the desired contradiction.

We can now move on to stronger theories.

Theorem: If T is any one of ACA_0^+, ATR_0, Π^1_1-CA_0 or Π^1_2-CA_0, $\mathcal{M} \vDash T$ and $\mathcal{U}$ is $\mathcal{M}$-generic for $\mathbb{P}_{I\mathcal{U}}$ then $\mathcal{M}^{\mathcal{U}} \vDash T^{I\mathcal{U}}$.

Corollary: For T in the Theorem, $T^{I\mathcal{U}}$ is a conservative extension of T for all sentences of second order arithmetic.

ACA_0^+ is immediate from the previous Theorem. The other cases require proof.

Conservation Results

ATR_0 requires knowing that IHT is provable in ACA_0^+ and so there are $e, k \in \mathbb{N}$ such that for any instance X, $\Phi_e^{X^{\omega \cdot k}}$ is a homogeneous set.

Π^1_1-CA_0 and Π^1_2-CA_0 require knowing that they imply strong Σ^1_1 (Σ^1_2)-DC_0.

If we are interested essentially in the divide between second and third order methods, then we would like an analogous result for Z_2. The same proof does not work as the relevant instances of strong dependent choice are not provable in Z_2.

Nonetheless, our methods applied to theories satisfying $\exists X(V = L[X])$ (which implies a very strong form of DC) yield a conservation result for $Z_2^{I\mathcal{U}}$ over Z_2 for essentially all sentences of combinatorial interest.

Proposition: If $T = Z_2 + \exists X(V = L[X])$, $\mathcal{M} \vDash T$ and $\mathcal{U}$ is $\mathcal{M}$-generic for $\mathbb{P}_{I\mathcal{U}}$ then $\mathcal{M}^{\mathcal{U}} \vDash T^{I\mathcal{U}}$.
Corollary: $Z_2^{I\mathcal{U}}$ is Π^1_4-conservative over Z_2. (An absoluteness argument.)

Ramsey Ultrafilters

We follow the same course as for idempotent ultrafilters with the natural changes.
In place of $\mathcal{AU}5$ we have the following axiom saying that $\mathcal{U}$ is Ramsey:

$\mathcal{AU}5R$ $\forall X(\langle X^{[n]}\rangle$ partitions M into pairwise disjoint nonempty sets $\wedge\ \forall n(n+1 \notin \delta_{\mathcal{U}}(X)) \rightarrow \exists Z(0 \in \delta_{\mathcal{U}}(Z)\ \wedge\ \forall n|X^{[n]} \cap Z| = 1))$

The language is the same as before. $T^{R\mathcal{U}}$ is the same as $T^{I\mathcal{U}}$ except that we replace $\mathcal{AU}5$ by $\mathcal{AU}5R$. The notion of forcing $\mathbb{P}_{R\mathcal{U}}$ and the associated definitions is the same as $\mathbb{P}_{I\mathcal{U}}$ except that the only restrictions on the U_i making up a condition u are that the U_i are infinite and nested with empty intersection.

All the same results hold with simplifications of some of the proofs as we no longer need to guarantee the extra requirements for conditions being in $\mathbb{P}_{I\mathcal{U}}$. Note that we only need ACA_0 as our base theory rather than IHT.

Conservation Results

Theorem: If $\mathcal{M} \vDash ACA_0$ and $\mathbb{U}$ and the associated $\mathcal{U}$ are $\mathcal{M}$-generic for $\mathbb{P}_{R\mathcal{U}}$ then $\mathcal{M}^{\mathcal{U}}$ is an $\mathcal{L}^{\mathcal{U}}$ structure and a model of $\mathrm{ACA}_0^{R\mathcal{U}}$. Moreover, any sentence Φ of $\mathcal{L}^{\mathcal{U}}$ (with second order parameters) is true in $\mathcal{M}^{\mathcal{U}}$ if and only if there is a $u \in \mathbb{U}$ such that $u \Vdash \Phi$. Finally, $\mathcal{U}$ is a Ramsey$\mathcal{M}$-ultrafilter.

Theorem: T is any one of ACA_0, ACA_0^+, ATR_0, Π^1_1-CA_0, Π^1_2-CA_0 or $Z_2 + \exists X(V = L[X]$, $\mathcal{M} \vDash T$ and $\mathcal{U}$ is $\mathcal{M}$-generic for $\mathbb{P}_{R\mathcal{U}}$ then $\mathcal{M}^{\mathcal{U}} \vDash T^{R\mathcal{U}}$.

Corollary: For T any of the theories mentioned above, $T^{R\mathcal{U}}$ is a conservative extension of T for all sentences of second order arithmetic. $\mathrm{Z}_2^{R\mathcal{U}}$ is Π^1_4-conservative over Z_2.