Geometric Group Theory, Genericity and the Theory of Computability

Paul Schupp

University of Illinois

Dedicated to Rod Downey

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Dehn

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The fundamental group of a compact orientable surface of genus $g \geq 2$ has a presentation

$$S_g = \langle x_1, ..., x_{2g}; x_1 ... x_{2g} x_1{}^{-1} ... x_{2g}{}^{-1} \rangle$$

Let R be the set of all cyclic permutations of the defining relator and its inverse. All elements of R are equal to the identity in S_g. Dehn proved that if $w = 1$ in S_g then w contains more than half of an element of R.

This gives Dehn's Algorithm for the word problem: If w contains more than half of $r \in R$, $r \equiv st$ with $|s| > |t|$ then replace s by t^{-1}. Note that this is a linear time algorithm.

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Cayley defined the basic geometric object associated with a finitely generated group $G = \langle X; R \rangle$ in 1878. Namely, its Cayley graph $\Gamma(G)$.

The vertices are the elements of G. For each $y \in X \cup X^{-1}$ there is a directed edge labelled y from g to gy. (The graph depends on the set of generators.)

So for the free ablian group $\mathbb{Z}^2$, the graph is the grid connecting integer lattice points in the plane.

A word w in the generators and their inverses is equal to the identity in the group exactly if w is the label of a closed loop α in Γ. So $\Gamma(G)$ contains all possible information about the word problem of G.

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Dehn was able to prove his theorem because S_g is a Fuchsian group, a discrete group of isometries of the hyperbolic plane $\mathbb{H}^2$ and its entire Cayley graph Γ is embedded in $\mathbb{H}^2$ as a regular tessellation of the plane by $4g$-gons.

Dehn calculated the curvature around closed curves α and concluded that α must contain the consecutive boundary of a $4g$-gon with at most three edges missing.

Dehn's paper is twenty-four years before the theory of computability and fifty-nine years before $\mathcal{NP}$-completness. (A case may be made that computational complexity begins with Lamé's 1844 theorem that the number of steps in the Euclidean algorithm is bounded by five times the number of decimal digits in the smaller number.)

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After Turing's 1936 paper and his proof of the unsolvability of the Halting Problem, other problems could be proved undecidable. A Turing machine can be viewed as a rewriting system and is thus "close" to a semigroup.

In 1947 Post and Markov independently constructed finitely presented semigroups with unsolvable word problem, In 1950 Turing himself constructed a finitely presented cancellation semigroup with unsolvable word problem.

In the 1950's Novikov and Boone independently constructed finitely presented groups with unsolvable word problem. Their constructions heavily used ideas from the papers on semigroups with unsolvable word problem.

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Of course, many groups have solvable word problem. We shall see that, indeed, *most* groups have solvable word problem.

Definition

A finite presentation $G = \langle X; , R \rangle$ is a *Dehn presentation* if the following holds: If w is a freely reduced word on the group alphabet $X \cup X^{-1}$ and $w = 1$ in G, then w contains more than half of an element of R.

So a Dehn presentation presents a group for which Dehn's algorithm solves the word problem.

In the 1960's and 70's, the quest for applying Dehn's algorithm to wider classes of groups gave rise to *small cancellation theory*. This was started by Tartakovskii in 1949, clarified by Greendlinger (1960) and then given a clear geometric explanation by Lyndon in 1966.

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Let $G = \langle X; R \rangle$ be a finitely presented group where we assume that R is *symmetrized*, that is, R is closed under taking cyclic permutations and inverses.

Definition

The symmetrized set R satisfies the condition $C'(1/6)$ if for every pair of elements $r_1, r_2 \in R$, with $r_2 \neq {r_1}^{-1}$ if $r_1 \equiv u_1 s$ and $r_2 \equiv s^{-1} u_2$ then $|s| < \frac{1}{6}|r_1|$.

As an example, consider the symmetrized set R generated by the defining surface group relator

$$x_1 x_2 x_3 x_4 {x_1}^{-1} {x_2}^{-1} {x_3}^{-1} {x_4}^{-1}$$

It is easy to check that at most one letter can cancel so R satisfies $C'(1/6)$.

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The basic theorem of small cancellation theory is

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(Greendlinger) If $G = \langle X; R \rangle$ satisfies $C'(1/6)$ then this presentation is a Dehn presentation.

Very few groups have planar Cayley graphs. One would like to imitate Dehn's argument but upon what structure is that possible? The Cayley graph contains all possible information about the word problem.

It turns out that the information about how one particular reduced word w can be equal to a particular product $c_1 r_{i_1} c_1{}^{-1} \ldots c_k r_{i_k} c_k{}^{-1}$ of conjugates of defining relators in the free group $\langle X \rangle$ can always be represented by a planar diagram Δ whose regions are labelled by elements of R and whose boundary is labelled by w.

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This was independently discovered by Lyndon in 1966 and it is immediately clear that the small cancellation condition says that adjacent regions can have only a small part of their boundaries in common. Calculating the curvature around the boundary of Δ shows that this boundary contains (as a consecutive part) more than half of a defining relator.

It turns out that such diagrams were discovered by van Kampen in 1933 but no use was made of them until Weinbaum, also in 1966, discovered van Kampen's paper.

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The $C'(1/6)$ condition is purely syntactic and it is easy to verify if a given presentation satisfies the condition. But there should be a geometric definition.

"What 'is' a small cancellation group? What is desired here is a geometric characterization of small cancellation groups. There should be a characterization of such groups ... by means of 'natural' geometric properties of their Cayley diagrams or in terms of their possible action on other complexes. Such a characterization would bring us back full circle to Dehn." (Schupp, 1973),

In his seminal 1987 monograph "Hyperbolic Groups", Gromov such a characterization. One can consider a Cayley graph as a locally compact geodesic metric space. Consider each edge as a copy of the unit interval of length 1 and define the distance between two points to be the length of a shortest path between them. Note that there may be several geodesics between two given points.

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Definition

(The Thin Triangle Condition) A geodesic metric space Y has *thin triangles* if there exists a constant δ such that, for any geodesic triangle in Y and any point p on one of the sides, p is within distance δ from some point on one of the other two sides.

Note that the Euclidean plane does not have thin triangles. Consider an equilateral triangle and the midpoint of one side as the length of the sides increases.

For the upper half-plane model of the hyperbolic plane $\mathbb{H}^2$, $\delta = log 3$.

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A finitely generated group $G = \langle X; R \rangle$ is *hyperbolic* if its Cayley graph $\Gamma(G)$ satisfies the Thin Triangle Condition.

The Cayley graph depends on the choice of generators but changing presentation just multiplies δ by a constant.

There are now several characterizations of hyperbolic groups, among which is

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A finitely generated group G is hyperbolic if and only if it has some Dehn presentation.

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Genericity first enters geometric group theory in the observation of Gromov, made precise by Olshanskii, that “most ” groups are hyperbolic.

Fix a finite generating set $X = \langle x_1, ..., x_k \rangle$ and fix a number d of defining relators. This is now called the ’few relators model”.

Let T_n denote the set of all presentations with generators X and d cyclically reduced defining relators of length at most n.

If $\mathcal{P}$ is a property of groups, let $\mathcal{P}_n$ denote the set of presentations in T_n which define groups with property $\mathcal{P}$.

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$$lim_{n \to \infty} \frac{|\mathcal{P}_n|}{|T_n|} = 1.$$

It is not difficult to show that presentations satisfying $C'(1/6)$ are generic and thus presentations generically define hyperbolic groups, for which the word problem is decidable in linear time.

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Is Worst-case Complexity a Good Measure of Difficulty?

The classes $\mathcal{P}, \mathcal{NP}$ and the general concept of being computable are worst-case measures: How hard are the most difficult instances? The example of the Simplex Algorithm shows that hard instances may be very rare.

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Density and Generic Sets of Elements

Fix a finite alphabet Σ and let Σ^* denote the set of all words over Σ. If $w \in \Sigma^*$, the *length* $|w|$ of w is the number of letters in w. Since we have a length function we can copy the classical definition of asymptotic density from number theory.

If $A \subseteq \Sigma^*$, then, for $n \geq 1$, the *density of A at n* is

$$\rho_n(A) = \frac{|\{w \in A : |w| \leq n\}|}{|\{w \in \Sigma^* : |w| \leq n\}|}$$

If $\rho(A) = \lim_{n\to\infty} \rho_n(A)$ exists then $\rho(A)$ is the *asymptotic density* of A. Of course, the limit does not exist in general.

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Definition

A is *generic* if $\rho(A) = 1$.

So in $\mathbb{N}$, the set of powers of 2 has density 0. Its complement has density 1 and is generic.

We say that a set *S* is *strongly generic* if

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Generic Computability and Complexity

Gurevich and Levin independently introduced the idea of *average-case complexity* as a better measure than worst-case complexity: Some problems with hard worst-cases may be much better on average.

But average-case is very hard to work with since it is very sensitive to the probability measure used and one must still consider the hardest cases. Kapovich, Miasnikov, Schupp and Shpilrain (2003) introduced *generic-case complexity* as a measure which is easier to work with than average-case complexity but still allows a nuanced analysis of problems where hard instances are very rare.

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Definition

Let S be a subset of Σ^* with characteristic function χ_S. A set S is *generically computable* if there exists a *partial computable* function Φ such that $\Phi(x) = \chi_S(x)$ whenever $\Phi(x)$ is defined (written $\Phi(x)\downarrow$) and the domain of Φ is generic in Σ^*.

We stress that *all* answers given by Φ must be correct but Φ need not be everywhere defined. .

We can "clock" Φ by requiring that on input w Φ answers in no more than $f(|w|)$ steps or we count Φ as not answering. In this case we say that Φ generically computes S in time f. We say that Φ *strongly generically computes* S in time f if the domain of Φ is strongly generic.

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An Example from Group Theory

In general, the word problem for finitely generated groups can be quite difficult, or indeed, not computable. But we have the following "Quotient Theorem".

Theorem

(Kapovich, Miasnikov, S., Shpilrain) Let G be a finitely generated group which has a subgroup H of finite index such that there is a homomorphism from H onto an infinite group K such that the word problem of K is solvable in time $\mathcal{C}$. Then the generic-case complexity of the word problem for G is generically time $\mathcal{C}$. If K contains a free group of rank 2, then the generic-case complexity of the word problem for G is strongly generically time $\mathcal{C}$.

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A beautiful theorem of Benjamin Baumslag and Steve Pride shows that every one-relator group with at least three generators has a subgroup of finite index which maps onto the free group of rank 2.

Hence, the word problem for *any* one-relator group with at least three generators is strongly generically linear time.

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(The Higman Embedding Theorem) A finitely generated group G is embeddable into a finitely presented group if and only if G has a presentation with a computably enumerable set of defining relators.

which was proved by Graham Higman in 1961. His motivation was to give the following proof that there is a finitely presented group with unsolvable word problem. Let $H = \langle a, b, t : ta^i ba^i t^{-1} = b^i ab^i, i \in S \rangle$ where S is a non-computable c.e. set with $0 \notin S$.

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$$H = \langle a, b, t : ta^i ba^i t^{-1} = b^i ab^i, i \in S \rangle$$

where $S \subseteq \mathbb{N} - \{0\}$ has the structure of an HNN extension of the free group $F = \langle a, b \rangle$.

The subgroups $A = gp\{a^i ba^i : i \in S\}$ and $B = gp\{b^i ab^i : i \in S\}$ are freely generated by the indicated generators in F and thus the map $\phi : a^i ba^i \to b^i ab^i$ is an isomorphism between A and B. The group H is obtained by adding a new generator t with the relations that conjugation by t sends A to B by the isomorphism ϕ.

Britton's Lemma says that a word w equals the identity in H exactly if w can be reduced to the empty word by free reductions and and *t-reductions* of the form $tut^{-1} = v$ and $t^{-1}vt = u$ where $u \in A$ and $v = \phi(u)$. Such a reduction reduces the number of t symbols.

So we see that the word problem for H is Turing equivalent to deciding membership in S for arbitrary S.

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The Higman Embedding Theorem preserves Turing degree. So for every non-zero c.e. degree $\boldsymbol{d}$, there is a finitely presented group with word problem of degree $\boldsymbol{d}$ but generically linear time.

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We do not know whether or not the isomorphism problem for one-relator presentations is solvable. However, Kapovich, Schupp and Shpilrain showed that it is strongly generically quadratic time. This again illustrates that it is possible to prove good generic-case results without knowing the worst-case complexity.

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The general idea of asymptotic geometry is that one can "see" certain features more clearly at a very large scale. For example, many of the features of a Gromov hyperbolic space are completely reflected in its boundary at infinity.

Remember that we are considering Cayley graphs as geodesic metric spaces. Let $G = \langle X; R \rangle$ be a finitely generated group and let d denote the word metric on its Cayley graph $\Gamma(G)$. Pick a point $p \in G$ as a basepoint, say the identity element. Let G_n be G but with metric d/n.

Let $\mathcal{A}$ be a metric ultraproduct of the G_n with respect to a non-principal ultrafilter. So $\mathcal{A}$ consists of all bounded sequences (x_i) where we identify two sequences (x_i) and (y_i) if $d_i(x_i, y_i) \to 0$.

$\mathcal{A}$ is called an *asymptotic cone* of G, and $\mathcal{A}$ formalizes the idea of "the group viewed from infinity".

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An $\mathbb{R}$-tree is a metric space such that between any two points x and y there is a unique geodesic $[x, y]$ and with the property that

$$[x, y] \cap [x, z] = \{x\} \Longrightarrow [x, y] \cup [x, z] = [y, z]$$

$\mathbb{R}^2$ with the standard metric is very far from being an $\mathbb{R}$-tree but $\mathbb{R}^2$ with the SNCF metric, the French railway metric, is an $\mathbb{R}$-tree

Pick a basepoint O. If two points are on a straight line through O their distance is the usual linear distance. If x and y are not on a straight line through O, then $d(x, y) = d(x, O) + d(O, y)$.

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So the fact that a finitely generated group has a presentation for which Dehn's Algorithm solves the word problem is equivalently reflected in the sturcture of its asymptotic cones.

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Theorem

(Gromov) A finitely generated group G is hyperbolic if and only if all its asymptotic cones are $\mathbb{R}$-trees.

So the fact that a finitely generated group has a presentation for which Dehn's Algorithm solves the word problem is equivalently reflected in the sturcture of its asymptotic cones.

While we are very far from anything remotely resembling this in computability theory, I think we begin to see hints in coarse computability.

Thank you!