The reverse mathematics of non-decreasing sequences

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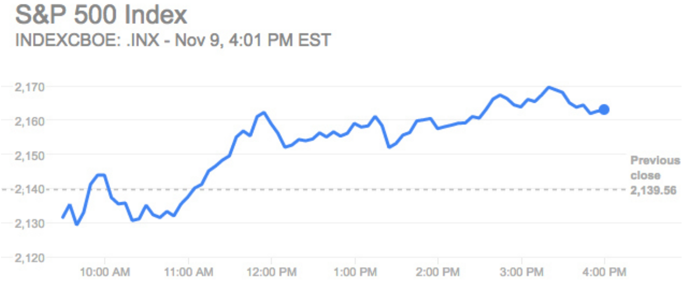

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Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable function such that

$$\forall x, s \quad f(x, s+1) \leq f(x, s)$$

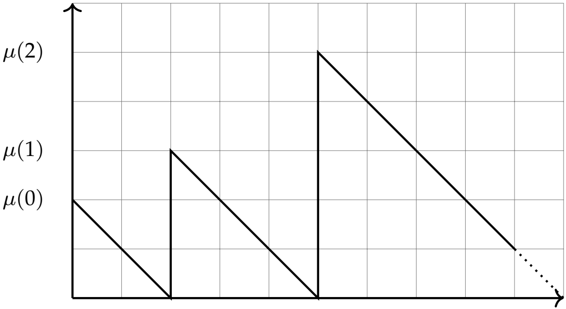
Let X be an infinite non-decreasing subsequence for

$$\tilde{f}(x) = \lim_s f(x, s)$$

How complicated must such an X be?

Identify the right abstraction of the problem

There is a Δ^0_2 function $g : \mathbb{N} \to \mathbb{N}$ for which every non-decreasing subsequence computes $\emptyset'$.



A function $g : \mathbb{N} \to \mathbb{N}$ is computably bounded if it is dominated by a computable function.

LNS

For every computable function such that $f(x, s+1) \leq f(x, s)$ there is a non-decreasing sequence for $\tilde{f}(x) = \lim_s f(x, s)$.

CNS

Every computably bounded Δ^0_2 function has an infinite non-decreasing sequence.

$$\tilde{f} \text{ is } \Delta^0_2 \text{ and dominated by } h(x) = f(x, 0).$$

Theorem (P.)

For every Δ^0_2 computably dominated function

- *there is a cone avoiding solution*
- *there is a solution which is low$_2$*
- *there is a solution computing no Martin-Löf random*

Corollary

$\mathsf{RCA}_0 + \mathsf{CNS} \nvdash \mathsf{WWKL}$

$$(F, X)$$

Initial segment Reservoir

- F is finite, X is infinite, $\max F < \min X$ (Mathias condition)
- $X \in \mathcal{M} \models \mathsf{WKL} \wedge \mathsf{D}^2_2$ (Weakness property)
- $\forall x \in X,\ F \cup \{x\}$ is non-decreasing (Combinatorics)

Forcing infinity

Instance : a Δ^0_2 function f dominated by h
Context : a condition (F, X)

- Pick $x \in X$
- Let $g(y) = \min(f(y), f(x))$
- Apply $\mathsf{D}^2_{f(x)+1}$ and get a set Y and a color c
- If $c < f(x)$, Y is our solution
- If $c = f(x)$, take $(F \cup \{x\}, Y)$

Forcing Σ^0_1 formulas

Instance : a Δ^0_2 function f dominated by h

Context : a condition (F, X) and a Σ^0_1 formula $\varphi(G)$

$$\mathcal{C} = \{g \text{ dom by } h : (\forall E \text{ non-decreasing } \subseteq X)\neg\varphi(F \cup E)\}$$

If $\mathcal{C} \neq \emptyset$

- Apply WKL and get $g \in \mathcal{C}$
- Get a non-decreasing subsequence $Y \subseteq X$ for g
- The condition (F, Y) forces $\neg\varphi(G)$

Forcing Σ^0_1 formulas

Instance : a Δ^0_2 function f dominated by h

Context : a condition (F, X) and a Σ^0_1 formula $\varphi(G)$

$$\mathcal{C} = \{g \text{ dom by } h : (\forall E \text{ non-decreasing } \subseteq X)\neg\varphi(F \cup E)\}$$

If $\mathcal{C} = \emptyset$

- In particular $f \notin \mathcal{C}$.
- Take $E \subseteq X$ non-decreasing for f such that $\varphi(F \cup E)$
- Using D^2_2 to obtain Y such that $(F \cup E, Y)$ is a condition

A function $g : \mathbb{N} \to \mathbb{N}$ is eventually increasing if for each $y \in \mathbb{N}$, the preimage of $\{y\}$ by g is finite.

LNS

For every computable function such that $f(x, s+1) \leq f(x, s)$ there is a non-decreasing sequence for $\tilde{f}(x) = \lim_s f(x, s)$.

ICNS

Every eventually increasing, computably bounded Δ^0_2 function has an infinite non-decreasing sequence.

If $\tilde{f}$ is not eventually increasing, it has a computable solution.

A function $g : \mathbb{N} \to \mathbb{N}$ is X-hyperimmune if it is not dominated by any X-computable function.

Theorem (P.)

Let $g_0, g_1, \dots$ be hyperimmune functions. For every eventually increasing, computably dominated Δ^0_2 function, there is a solution H such that the g's are H-hyperimmune.

Corollary

$\mathsf{RCA}_0 + \mathsf{ICNS} + \mathsf{EM} + \mathsf{WKL} \nvdash \mathsf{SADS}$

Theorem (Downey)

Every non-zero r.e. degree contains an r.e. set without the universal splitting property.

The strength of non-decreasing sequences

A function $g : \mathbb{N} \to \mathbb{N}$ is diagonally non-computable (DNC) if $g(e) \neq \Phi_e(e)$ for every $e \in \mathbb{N}$.

Theorem (Liang Yu)

There is a computable function satisfying $f(x, s+1) \leq f(x, s)$ such that every infinite non-decreasing sequence for $\tilde{f}$ computes a DNC function.

Proof: $f(x, s) =$ plain Kolmovorov complexity of x at stage s.

A function $g : \mathbb{N} \to \mathbb{N}$ is hyperimmune if it is not dominated by any computable function.

Theorem (P.)

There is a computable function satisfying $f(x, s+1) \leq f(x, s)$ such that every infinite non-decreasing sequence for $\tilde{f}$ computes a hyperimmune function.

Proof: by a finite injury priority argument.

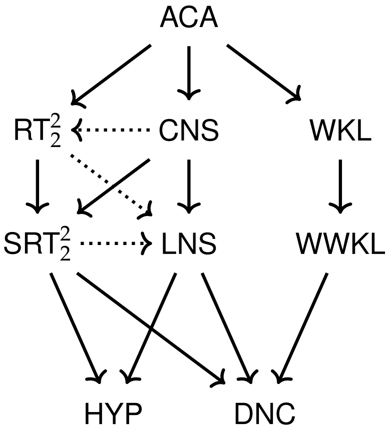
A function f is X-hypersurjective if there is an infinite $L \subseteq \mathbb{N}$ such that for every X-computable array $A_0, A_1, \dots$ and every $y \in L$, $f[A_i] = \{y\}$ for some $i \in \mathbb{N}$.

Theorem (P.)

Fix f hypersurjective. Every computable instance of WKL *and* RT^2_2 *has a solution H such that f is H-hypersurjective.*

Corollary

$\mathsf{RCA}_0 + \mathsf{RT}^2_2 + \mathsf{WKL} \nvdash \mathsf{CNS}$


ACA
RT^2_2
CNS
WKL
SRT^2_2
LNS
WWKL
HYP
DNC

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