My work with Rod 1995-2001

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Jan. 7, 2017

Degree structures

Back in the days before everyone started using the beamer package, degree structures were studied even more intensely than now.

My work with Rod 1995-2001 was centred around degree structures.

We looked at reducibilities other than Turing:

- $\leq_Q$,
- $\leq^p_m$, $\leq^p_T$ (polynomial time m- and Turing),
- $\leq_S$ (Solovay).

Three degree structures

Rod, I and co-authors studied three very different degree structures.

- The Q degrees of r.e. sets, with LaForte
- polynomial time many-one and Turing degrees of exponential time sets of strings
- Solovay degrees of left-r.e. reals, with Hirschfeldt.

I. Q-degrees of r.e. sets

Q-reducibility was defined by Tennenbaum (according to Rogers). The "Q" stands for "quasi". In the Turing case we have for r.e. sets $A \leq_T B \Leftrightarrow \exists R$ r.e. $\forall y\,[y \in \overline{A} \leftrightarrow \exists z[\langle y, z\rangle \in R \;\&\; D_z \subseteq \overline{B}]]$.

For Q-reducibility, the reduction procedure is allowed at most one negatively answered oracle question. Thus, each D_z is a singleton. Now let $W_{g(y)} = \bigcup_{\langle y,z\rangle \in R} D_z$ for computable g:

Definition 1 *For $A, B \subseteq \mathbb{N}$, let $A \leq_Q B \Leftrightarrow$*

$$\exists g \textit{ computable } \forall y\,[y \in A \leftrightarrow W_{g(y)} \subseteq B].$$

For r.e. sets $A \leq_Q B \Rightarrow A \leq_T B$.

Q-reducibility was used for a structural solution to Post's problem (Marchenkov), and for the complexity of word problems of groups.

Meets and joins in $\mathcal{R}_Q$

Results below are from Downey, N, LaForte,

Computably enumerable sets and quasi-reducibility,

APAL 95.1 (1998): 1-35.

First note that $\mathcal{R}_Q$ is an upper semilattice where the sup of degrees of A, B is the degree of $A \oplus B$.

Theorem 2 *There is a minimal pair of r.e. Q-degrees that have the same Turing degree.*

The proof uses a pinball machine model.

Theorem 3 *There is a meet-irreducible r.e. Q-degree outside any nontrivial upper cone.*

Density of $\mathcal{R}_Q$

The hardest result answers a question of Ishmukametov.

Theorem 4 *The r.e. Q-degrees are dense.*

Given r.e. $B <_Q A$, we want to build r.e. C such that $B <_Q B \oplus C <_Q A$. The proof is harder than in the Turing case. For instance, the usual permitting technique for $C \leq_T A$ doesn't yield a Q-reduction. We use a tree of strategies.

Truth table reducibility (not dense) and Q-reducibility are incomparable reducibilities on the r.e. sets; the result shows that Q-reducibility is in a sense closer to Turing.

Undecidability of $\mathrm{Th}(\mathcal{R}_Q)$

Theorem 5 *The first-order theory of $\mathcal{R}_Q$ is undecidable.*

This is proved by encoding with parameters any given computable partial order $(\mathbb{N}, \prec)$.

- The domain is represented by a Slaman-Woodin set $\{G_i\}$ (the sets below C of minimal degree cupping P above Q).
- Build a further parameter L with $i \preceq k \leftrightarrow G_i \leq_Q G_k \oplus L$.

Some later work on Q-degrees

- Affatato, Kent and Sorbi 2007: paper on s-degrees (singleton reducibility). This is a restricted version of e-reducibility.

  Note that $\overline{A} \leq_s \overline{B} \Leftrightarrow A \leq_Q B$.

  They show the Σ^0_2 and the Π^0_1 s-degrees are undecidable, using "exact degree theorems".

- Arslanov, Batureshin and Omanadze (2007) work on n-r.e. Q-degrees.

  They also prove that there is a noncappable incomplete r.e. Q-degree.

II: subrecursive degree structures

Let Σ be alphabet, $X, Y \subseteq \Sigma^*$ languages over Σ.

$X \leq_m^p Y \Leftrightarrow \exists f \in P\ [X = f^{-1}(Y)]$

$X \leq_T^p Y \Leftrightarrow\ \exists$ polynomial time bounded oracle TM which computes X with oracle Y.

Ladner (1975) proved that the degree structures induced on the computable languages are dense.

Theorem 6 (Downey and N., JCSS 2003) *The polynomial time many-one and Turing degrees of languages in* $\mathtt{DTIME}(2^n)$ *have an undecidable theory.*

Instead of 2^n, one can take any nondecreasing time constructible function $h : \omega \to \omega$ such that $P \subset \textsc{Dtime}(h)$. E.g. $h(n) = n^{\log \log n}$.

Undecidable ideal lattices

- A structure $(\mathbb{N}, \preceq, \wedge, \vee)$ is a Σ^0_k-boolean algebra if $\preceq$ is Σ^0_k, and the operations $\wedge, \vee$ are recursive in $\emptyset^{(k-1)}$.
- A Σ^0_k-boolean algebra $\mathcal{B}$ is called **effectively dense** if there is a function $F \leq_T \emptyset^{(k-1)}$ such that $\forall x \; [F(x) \preceq x]$ and $\forall x \not\equiv 0 \;\; [0 \prec F(x) \prec x]$.
- For a Σ^0_k–boolean algebra $\mathcal{B}$, let $\mathcal{I}(\mathcal{B})$ be the lattice of Σ^0_k–ideals of $\mathcal{B}$ with $\cap$ and $\vee$ as operations.

Theorem 7 (N, Trans. AMS 2000) *Suppose $\mathcal{B}$ is an effectively dense Σ^0_k-Boolean algebra. Then Th*$(\mathbb{N}, +, \cdot) \equiv_m \mathrm{Th}(\mathcal{I}(\mathcal{B}))$.

It is much easier to show that $\mathrm{Th}(\mathcal{I}(\mathcal{B}))$ is hereditarily undecidable [N., Bull. LMS 1997].

Undecidability via coding $\mathcal{I}(\mathcal{B})$

It is often natural to interpret with parameters $\mathcal{I}(\mathcal{B})$ in a structure. This shows that the structure has an undecidable theory.

If no parameters are needed, it yields an interpretation of $\mathrm{Th}(\mathbb{N}, +, \cdot)$ in the theory of the structure.

- Intervals of $\mathcal{E}^*$ that are not Boolean algebras; no parameters needed (N., 1997)
- Computable sets with parameterized reducibilities (Coles, Downey, Sorbi, year?).
- Solovay degrees of left-r.e. reals (Downey, Hirschfeldt and LaForte, JCSS, 2007). Details later.

Supersparse sets in complexity theory

We will apply the method of coding $\mathcal{I}(\mathcal{B})$ in the proof that polytime degrees of $\textsc{Dtime}(h)$ have an undecidable theory. Here $k = 2$, because $\leq_m^p$ and $\leq_T^p$ are Σ_2^0-relations on such a class.

Definition 8 (Ambos-Spies 1986) *Let $f\colon \omega \to \omega$ be a strictly increasing, time constructible function. We say that a language $A \subseteq \{0^{f(k)} : k \in \omega\}$ is* super sparse *via f if*

" $0^{f(k)} \in A$?" can be determined in time $O(f(k+1))$.

Supersparse sets exist in the time classes we are interested in.

Lemma 9 (Ambos-Spies 1986) *Suppose that $h : \omega \to \omega$ is a nondecreasing time constructible ("nice") function with $P \subset \textsc{Dtime}(h)$. Then there is a super sparse language $A \in \textsc{Dtime}(h) - P$.*

Interpreting $\mathcal{I}(\mathcal{B})$ in $[\mathbf{o}, \boldsymbol{a}]$ for super sparse $\boldsymbol{a}$

Now let f, h be time constructible functions as above, let $A \in \textsc{Dtime}(h) - P$ be supersparse via f, and $\boldsymbol{a}$ be its degree. Ambos-Spies has shown that $[\mathbf{o}, \boldsymbol{a}]$ is a distributive lattice that does not depend on the reducibility.

- Each complemented element in $[\mathbf{o}, \boldsymbol{a}]$ is the degree of a splitting $A \cap R$, where R is polytime.
- This implies that the algebra $\mathcal{B}$ of complemented elements is Σ^0_2 and effectively dense.
- Downey and N. showed that for each ideal I in $\mathcal{I}(\mathcal{B})$, there is $\boldsymbol{c}_I \leq \boldsymbol{a}$ such that $\boldsymbol{x} \in I \Leftrightarrow \boldsymbol{x} \leq \boldsymbol{c}_I$ for each $\boldsymbol{x} \leq \boldsymbol{a}$.

So one can interpret $\mathcal{I}(\mathcal{B})$ in $[\mathbf{o}, \boldsymbol{a}]$ without parameters (and hence in $\textsc{Dtime}(h)$ with parameter $\boldsymbol{a}$).

III. Solovay reducibility on left-r.e. reals

A real α is **left-r.e.** if there is a non-decreasing effective sequence (α_s) of rationals converging to α.

We will use α, β, γ to denote left-r.e. reals. We think of them as equipped with an effective sequence of rationals of this kind.

Example of a left-r.e. real: The halting probability of a fixed universal prefix-free machine U

$$\Omega = \sum\{2^{|\sigma|} : U(\sigma)\downarrow\}.$$

Solovay reducibility

Solovay (1975) introduced a reducibility $\leq_S$ to compare the "randomness content" of left-r.e. reals.

$\beta \leq_S \alpha \iff$

$$\exists C \in \mathbb{Q}\ \exists f \text{ computable increasing} \quad \forall s\ [\beta - \beta_{f(s)} \leq C(\alpha - \alpha_s)].$$

He proved that $\beta \leq_S \alpha \Rightarrow \exists c\ \forall n\ K(\beta \upharpoonright n) \leq K(\alpha \upharpoonright n) + c$.

Ω is $\leq_S$–complete. Kucera and Slaman (2001) showed that, for left-r.e. reals,

$$\leq_S\text{–complete} \Leftrightarrow \text{ML-random}.$$

Fact 10 $\beta \leq_S \alpha \iff \exists C$ *rational* $\exists \gamma\ C(\beta + \gamma) = \alpha$.

Fact 11 $\beta \leq_S \alpha \Rightarrow \beta \leq_T \alpha$. *But* $\leq_S$ *is incomparable with* $\leq_{wtt}$.

Algebraic properties

Results below are from Downey, Hirschfeldt, Nies, *Randomness, computability, and density*, Siam J. Computing 31, 2002.

Let $\mathcal{S}$ be the degree structure induced on the r.e. reals. We investigate the algebraic properties of $\mathcal{S}$.

$\mathcal{S}$ is an upper semilattice (u.s.l.) where the sup is given by the usual addition.

Recall that an u.s.l. is distributive if it satisfies

$$a \leq b \vee c \Rightarrow a = \widetilde{b} \vee \widetilde{c} \text{ for some } \widetilde{b} \leq b, \widetilde{c} \leq c.$$

Proposition 12 *$\mathcal{S}$ is a distributive u.s.l.*

Among the common degree structures on r.e. sets, $\mathcal{R}_m$ (many-one) and $\mathcal{R}_{wtt}$ (weak truth-table) are distributive.

Is $\mathcal{S}$ more like $\mathcal{R}_{wtt}$, or more like $\mathcal{R}_m$?

Density

Using standard coding and preservation strategies, we obtain upward density.

Theorem 13 *Let $\gamma <_S \Omega$. Then there is β such that $\gamma <_S \beta <_S \Omega$.*

If $\alpha <_S \Omega$, we prove that in a sense any sequence for Ω converges much slower than one for α. This gives combined splitting and density below α.

Theorem 14 *Let $\gamma <_S \alpha <_S \Omega$. There are β^0 and β^1 such that $\gamma <_S \beta^0, \beta^1 <_S \alpha$ and $\beta^0 + \beta^1 = \alpha$.*

Combining the two, we obtain a (non-uniform) proof of density.

$\mathcal{S}$ shares this property with $\mathcal{R}_{wtt}$.

Random left-r.e. reals

Fact 15 *If one of α, β is ML-random, then $\gamma = \alpha + \beta$ is ML random.*

By contraposition suppose that γ is not ML-random. So $\gamma \in \bigcap G_m$ for a ML-test (G_m), where $\lambda G_m \leq 2^{-m-1}$. Build a ML-test (H_m) for α: At stage s, if $\gamma_s \in I$ where $I = [x, y)$ is a maximal subinterval of $G_{m,s}$, then put the interval

$$J = [x - \beta_s - (y - x), y - \beta_s]$$

into H_m. (Note that J is twice as long as I.)

A similar fact *fails* for left-r.e. reals and weaker randomness notions. The opposite was announced (wrongly) during the talk.

Fact 16 (with Miyabe and Stephan, 2017) *There is α partial computably random and β such that $\alpha + \beta$ is not Kurtz random.*

Random left-r.e. reals

Now for the converse for ML-randomness.

Theorem 17 *If $\alpha + \beta$ is ML-random, then one of α, β is ML random.*

(Several years after our paper appeared in 2001, Kucera pointed out that this was claimed without proof by Demuth[a].

Using the Kucera and Slaman Theorem that any random left-r.e. real is $\leq_S$-complete, this implies

Corollary 18 *In $\mathcal{S}$, the greatest element is join irreducible.*

$\mathcal{S}$ shares this property with $\mathcal{R}_m$.

[a] Constructive pseudonumbers, Comment. Math. Univ. Carolinae, vol. 16 (1975), pp. 315 - 331, Russian)

Later work

- Downey, Hirschfeldt and LaForte, *Undecidability of the structure of the Solovay degrees of c.e. reals* (2002) uses the method of coding $\mathcal{I}(\mathcal{B})$. Similar to the complexity case, they build an r.e. set A such that all complemented elements below are given by r.e. splittings.

- Downey, Hirschfeldt and LaForte, *Randomness and reducibility*, JCSS, 2007 (also D-H book): proof of results such as density in a more general axiomatic setting; works for $\leq_S, \leq_C, \leq_K, \leq_{rK}$ but not $\leq_{sw}$.

- Barmpalias, Bull. Symb. Log. 19(3), 2013:
  elementary differences between the structures etc.

Additive cost functions

Not a lot has happened on the structure of Solovay degrees in recent years. However, I used $\leq_S$ in the paper "Calculus of cost functions" (to appear in "The Incomputable").

For a r.e. real β with a given approximation let $\boldsymbol{c}_\beta(x,s) = \beta_s - \beta_x$.

Proposition 19 *$\boldsymbol{c}_\alpha$ implies $\boldsymbol{c}_\beta$ for some approximations of α, β*

$$\Leftrightarrow \beta \leq_S \alpha.$$

E.g. $\boldsymbol{c}_\Omega$ is the strongest additive cost function. Obeying it characterises the K-trivials.

Question 20 *Find β such that the Δ^0_2 sets obeying $\boldsymbol{c}_\beta$ form a proper Turing ideal different from the K-trivials.*

More open questions

Question 21 *Do the degree structures considered above interpret true arithmetic?*

Question 22 *Suppose* $\boldsymbol{a} \neq \mathbf{o}$ *is a polytime* m *(or Turing) degree. Is* $\mathrm{Th}[\mathbf{o}, \boldsymbol{a}]$ *undecidable?*

Question 23 *How can we distinguish incomplete Solovay degrees of left-r.e. reals? For instance, are there two non-isomorphic initial segments strictly below* Ω*?*

Also: study the Solovay degrees of left-r.e. Schnorr randoms.