Computable linear orders

Keng Meng Ng

Nanyang Technological University, Singapore

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Motivating questions

- Study how computation interacts with various mathematical concepts.
- Measure how regular an algebraic object is by automorphisms.
- Rigidity / Symmetry.
- We're going to look at *computable* linear orders and their *effective* automorphisms.

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- Refining the classical notion: the number of automorphisms of a structure is the same in every copy.
- Invariance no longer holds if we look at the number of effective automorphisms of different computable copies of $\mathcal{A}$.

  Example: It is easy to construct a computable copy of $(\mathbb{Z}, <)$ with no computable automorphism (other than *id*). Yet $\mathbb{Z}$ is (classically) not rigid.
- It's better to quantify over computable copies.

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Automorphisms in simple examples

- Let's look at the nicest linear order, $\mathbb{Q}$, denoted by η.
- (Remmel) A linear order is computably categorical iff it has finitely many successivities.
- What automorphisms does each copy of η have?
  - Each computable copy of η has a nontrivial computable automorphism.
  - In fact, each automorphism F of η is also strongly nontrivial in the sense that for some x, the interval $(x, F(x))$ is infinite.
- Obviously, this is true as well for any $\mathcal{L}$ which contains an η-interval.

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Automorphisms in simple examples

- The two examples we have:

  If $\mathcal{L}$ contains an η-interval: *Every* computable copy of $\mathcal{L}$ has a (strongly) nontrivial computable automorphism.

  If $\mathcal{L} \cong \mathbb{Z}$: $\mathcal{L}$ has a computable copy with no computable automorphisms.

Theorem (Schwartz)

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Π^0_1-rigidity

- Computable rigidity is thus completely classified. What else can we say?

Definition

We say that $\mathcal{L}$ is Π^0_1-rigid if there is a computable copy $\mathcal{A} \cong \mathcal{L}$ such that $\mathcal{A}$ has no strongly nontrivial Π^0_1-automorphism.

- Note that Σ^0_1-rigidity is the same as computable-rigidity.
- Note also that $\mathcal{L}$ has no $\mathbb{Z}$-interval if and only if every non-trivial automorphism is strongly non-trivial.

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Kierstead's Conjecture

- Kierstead (1987) investigated the very similar $\mathcal{L} \cong 2 \cdot \eta$.
- By Schwartz's criterion, $k \cdot \eta$ is computably-rigid.
- Obviously, $k \cdot \eta$ is *not* Δ^0_2-rigid.
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Conjecture (Kierstead 1987)

$\mathcal{L}$ is Π^0_1-rigid if and only if $\mathcal{L}$ does not contain an η-interval.

- Note that the conjecture is obviously false if we do not require the automorphism to be strongly nontrivial. For example, every copy of $\mathbb{Z}$ has a Π^0_1 non-identity automorphism $x \mapsto S(x)$.
- Kierstead verified his conjecture for the case $\mathcal{L} \cong 2 \cdot \eta$.
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η-like linear orders

- Cooper, Harris and Lee verified the conjecture for a large subclass of η-like linear orders.
- $\mathcal{L}$ is η-like if

$$\mathcal{L} \cong \sum \{F(q) \mid q \in \mathbb{Q}\}$$

  for some function $F : \mathbb{Q} \mapsto \mathbb{N} \setminus \{0\}$. Call F block function for $\mathcal{L}$.
- Since every block is finite, every non-identity automorphism is strongly nontrivial.
- These linear orders are useful in testing general properties of linear orders.
- (McCoy) Any linear order with no interval of type ω or ω^* is η-like (except for finitely many points).

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η-like linear orders

- (Folklore) If $\mathcal{L}$ is η-like and computable, then clearly we can choose a block function $F \leq_T \emptyset''$.
- (Frolov, Zubkov) It is easy to see that if F is $\mathbf{0}'$-limitwise monotonic, then there is a computable $\mathcal{L}$ with block function F.
  - F is $\mathbf{0}'$-limitwise monotonic if there is a $g \leq_T \mathbf{0}'$ such that $F(n) = \lim_s g(n, s)$ and g is non-decreasing in s.
- (Harris) On the other hand every η-like with no strongly η-like subinterval $\mathcal{L}$ has a $\mathbf{0}'$-limitwise monotonic block function.

Theorem (Cooper, Harris, Lee)

Every η-like linear order with a $\mathbf{0}'$-limitwise monotonic block function and no η-interval is Π^0_1-rigid.

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Extensions of Cooper, Harris, Lee

- Wu and Zubkov extended this result to the class of linear orders of order-type

$$\sum \{ F(q) \mid q \in \mathbb{Q} \}$$

where $F : \mathbb{Q} \mapsto \mathbb{N} \cup \{\zeta\} \setminus \{0\}$ is $\mathbf{0'}$-limitwise monotonic. Here ζ represents the ordering $\mathbb{Z}$ and $\zeta > n$ for every $n \in \mathbb{N}$.

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Refuting Kierstead's Conjecture

- However, Kierstead's Conjecture is false:

Theorem (N, Zubkov)

There is a computable linear order with no η-intervals and is not Π^0_1-rigid.

- The linear order constructed is not η-like, but has order type $\sum\{F(q) \mid q \in \mathbb{Q}\}$, where $F : \mathbb{Q} \mapsto \mathbb{N} \setminus \{0\}$ is a partial $\mathbf{0'}$-limitwise monotonic function. Here $F(q)\uparrow$ stands for the order-type $\mathbb{Z}$.

Refuting Kierstead's Conjecture

- First prove a uniform version: Given $\mathcal{L}^*$ we construct $\mathcal{L}$ with no η-interval and φ such that either $\mathcal{L} \not\cong \mathcal{L}^*$ or φ is a (strongly nontrivial) Π^0_1-automorphism of $\mathcal{L}^*$.
- Some issues:
  1. Since $\mathcal{L}^*$ might not be Δ^0_2-categorical there is no hope of guessing for an approximation to an isomorphism $\mathcal{L} \mapsto \mathcal{L}^*$. The trick then is to make $\mathcal{L}$ look like $k \cdot \eta$ while waiting for block sizes in $\mathcal{L}^*$ to go down.
  2. To make φ strongly nontrivial.
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- Notice that if $\mathcal{L}^* \cong \mathcal{L}$ then we end up making $\mathcal{L} \cong \mathbb{Z} \cdot \eta$. Otherwise if $\mathcal{L}^* \ncong \mathcal{L}$ then we end up making $\mathcal{L} \cong k \cdot \eta$ for some $k \in \mathbb{N}$.
- Thus $\mathcal{L}$ has strongly η-like intervals (this is necessary, otherwise $\mathcal{L}$ will have a $\mathbf{0}'$-limitwise monotonic block function).
- For the full construction of $\mathcal{L}$, put all the different requirements together using separators.
- Some additional effort needed to keep the different modules from interacting.

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Summary of Π^0_1-rigidity

Order type (no η-interval)	Π^0_1-rigid
Discrete	✓ (Downey, Moses)
Blocks of a single finite size (strongly η-like)	✓ (Kierstead)
Blocks of finite size (η-like) with $\mathbf{0}'$-l.m.f. block function	✓ (Cooper, Harris, Lee)
Blocks of finite size or type $\mathbb{Z}$ with $\mathbf{0}'$-l.m.f. block function	✓ (Wu, Zubkov)
Blocks of finite size or type $\mathbb{Z}$ with partial $\mathbf{0}'$-l.m.f. block function	× (N, Zubkov)

Questions

- Since not every η-like linear order has a $\mathbf{0}'$-l.m.f. block function, does Kierstead's conjecture hold for η-like linear orders?

Theorem (N, Zubkov)

There is a computable η-like $\mathcal{L}$ with no η-interval such that for every computable $\mathcal{L}' \cong_{\Pi^0_2} \mathcal{L}$ has a strongly nontrivial Π^0_1 automorphism.

- What if we allow blocks to be either finite or type ω or ω^*?
- Kierstead's conjecture is related to the so-called "*self-embedding conjecture*": Every copy of $\mathcal{L}$ has a nontrivial computable self-embedding if and only if $\mathcal{L}$ contains a strongly η-like interval.
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