On Σ–presentability of some structures of analysis over hereditarily finite superstructures

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$\mathbb{HF}(\mathfrak{M})$: a "programmer's" universe over basic data type $\mathfrak{M}$

$$\begin{aligned}
\mathbb{HF}_0(\mathfrak{M}) &= \mathfrak{M} \\
\mathbb{HF}_{n+1}(\mathfrak{M}) &= \mathbb{HF}_n(\mathfrak{M}) \cup S_{<\omega}(\mathbb{HF}_n(\mathfrak{M})) \\
\mathbb{HF}(\mathfrak{M}) &= \bigcup_{n<\omega} \mathbb{HF}_n(\mathfrak{M})
\end{aligned}$$

$\mathbb{HF}(\mathfrak{M})$: all sets which could be explicitly defined using $\{,\}$, $\varnothing$, and elements of $\mathfrak{M}$, for instance, if $\mathfrak{M} = \mathbb{R}$: $\varnothing$, $\{\varnothing\}$, $\{1, \sqrt{2}\}$, $\{r, \varnothing, \{u, \pi, \{\varnothing, \{7\}\}\}$, $\sqrt[17]{29}\}$, $r, u \in \mathbb{R}$, all finite ordinals, etc. ...

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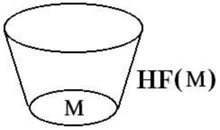
Specific membership relation '$\in$' (elements of $\mathfrak{M}$ are urelements):

$x \in y \;\Leftrightarrow\; (x \in \mathbb{HF}(\mathfrak{M})) \wedge (y \in \mathbb{HF}(\mathfrak{M}) \setminus \mathfrak{M}) \wedge (x \text{ is a member of } y)$

- $U(x) \;\Leftrightarrow\; x \in M$
- Language for $\mathbb{HF}(\mathfrak{M})$:

  $\langle U, \in, \varnothing, \langle \text{Predicates of } \mathfrak{M} \rangle \rangle$



Σ–definability vs classical computability

Σ–formulas: a special class of formulas that defines ‘computably enumerable sets’ in $\mathbb{HF}(\mathfrak{M})$

- computably enumerable $\sim$ definable by Σ–formulas with parameters (Σ–definable)
- computable $\sim$ Δ–definable ($=$ Σ– and $\neg\Sigma$–definable)

'Computable model theory over admissible sets':

Computable structures $\rightarrow$ Σ–presentable structures

Definition (Yu. L. Ershov)

*If each element of a structure $\mathfrak{A}$ is associated with a set of its codes (notations) from an admissible set $\mathbb{A}$ so that the diagram of this structure is a Σ–definable subset of $\mathbb{A}$, then such a coding is called a Σ–*presentation *of $\mathfrak{A}$.*

If each element of $\mathfrak{A}$ is coded by a single element of $\mathbb{A}$ then such a presentation is called *simple*.

Definition

An element $a \in \mathfrak{M}$ is called $\exists$–algebraic over $A \subseteq \mathfrak{M}$ *if there exists an $\exists$–formula $\varphi(x, \bar{y})$ and parameters $\bar{b} \in A$ such that the set $\varphi^{\mathfrak{M}}[x, \bar{b}]$ is finite and contains a.*

$\mathbf{C}_{\exists}^{\mathfrak{M}}(A)$: the set of all algebraic elements over A.

Definition

A structure $\mathfrak{M}$ is called $\exists$–Steinitz if $\mathbf{C}_{\exists}^{\mathfrak{M}}(A)$ has the following exchange property:
$a \in \mathbf{C}_{\exists}^{\mathfrak{M}}(A \cup \{b\}) \setminus \mathbf{C}_{\exists}^{\mathfrak{M}}(A)$, *implies* $b \in \mathbf{C}_{\exists}^{\mathfrak{M}}(A \cup \{a\})$.

Examples: $\mathbb{R}$, $\mathbb{C}$, any model of strongly minimal model complete theory, any model complete field or ordered field, algebraically closed field, etc.

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Definition

A set $X \subseteq \mathfrak{M}$ *is called independent if for all* $x \in X$ *holds* $x \notin \mathbf{C}_{\exists}^{\mathfrak{M}}(X \setminus \{x\})$.

Definition

The dimension of a set*: the cardinality of any its maximal independent set.*

Theorem (M., 2014)

Assume that $\mathfrak{M}$ is an $\exists$–Steinitz structure of a finite signature. Let $\mathfrak{A}$ be an arbitrary structure of a finite signature for which there exist a family $(F_i)_{i<\omega}$ of unary operations definable by terms with parameters and a family $(A_i)_{i<\omega}$ of subsets of $\mathfrak{A}$ such that:

1. *all the sets $F_i\,[A_i]$ are uncountable*
2. *for any sequence $(a_i)_{i<\omega} \in \prod_{i<\omega} A_i$ there is an element $b \in \mathfrak{A}$ such that for all $i < \omega$ holds $F_i(b) = F_i(a_i)$.*

Then $\mathfrak{A}$ is cannot be embedded into a structure having a simple Σ–presentation over $\mathbb{HF}(\mathfrak{M})$ with parameters.

Proof: having assumed that such a presentation exists we arrive at a contradiction like 'a finite set has infinite dimension'.

Theorem (M., 2014)

The following structures are not embeddable into structures having simple Σ–presentations with parameters over $\mathbb{HF}(\mathfrak{M})$ where $\mathfrak{M}$ is an $\exists$–Steinitz structure of a finite signature:

1. $P(\omega)$, $P(\omega)/_{Fin}$
2. *The lattice of all open (closed) subsets of* $\mathbb{R}^n$, $n > 0$
3. $\mathrm{Sym}\,(\omega)$, $\mathrm{Sym}\,(\omega)/_{Fin}$
4. *The group (semigroup) of all permutations (mappings) Σ–definable with parameters over* $\mathbb{HF}(\mathbb{R})$
5. *The semigroup* ω^ω

Theorem

The following structures are not embeddable into structures having simple Σ–presentations with parameters over $\mathbb{HF}(\mathfrak{M})$ where $\mathfrak{M}$ is an $\exists$–Steinitz structure of a finite signature:

1. $\mathtt{Aut}\ \langle \mathbb{Q}, < \rangle$: *the automorphism group of the ordering on the rational numbers in the signature with a single operation of composition.*
2. $\mathtt{Aut}\ \langle \mathbb{R}, < \rangle$: *the automorphism group of the ordering on the reals in the signature with a single operation of composition.*
3. $\mathrm{C}(\mathbb{R}^n)$: *the semigroup of all continuous mappings from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$, *for any* $n > 0$.
4. $\mathrm{C}^1(\mathbb{R}^n)$: *the semigroup of all continuously differentiable functions from* $\mathbb{R}^n$ $\mathbb{R}^n$, *for any* $n > 0$.

Theorem (A new sufficient condition for nonpresentability)

Assume that $\mathfrak{M}$ *is an* $\exists$*–Steinitz structure of a finite signature. Let* $\mathfrak{A}$ *be an arbitrary structure of a finite signature such that there exist a family* $(F_i)_{i<\omega}$ *of unary partial functions* **definable by** Σ**–formulas with parameters over** $\mathbb{HF}(\mathfrak{A})$ *and a family* $(A_i)_{i<\omega}$ *of subsets of* $\mathfrak{A}$ *such that:*

1. *for each* $i < \omega$ *holds* $A_i \subseteq \mathtt{dom}(F_i)$, *and* $F_i[A_i]$ *is uncountable*
2. *for any sequence* $(a_i)_{i<\omega} \in \prod_{i<\omega} A_i$, *there exists a* $b \in \mathfrak{A}$ *such that for all* $i < \omega$ *holds* $F_i(b) = F_i(a_i)$.

Then $\mathfrak{A}$ **has no simple** Σ**–presentations over** $\mathbb{HF}(\mathfrak{M})$ **with parameters**.

Theorem

Let D be an arbitrary nonprincipal filter over ω and $\mathfrak{M}$ be an $\exists$–Steinitz structure of a finite signature. Then the filtered power $\langle \mathbb{R}^\omega/D, \mathtt{St}, \mathtt{Inf} \rangle$ extended with unary predicates $\mathtt{St}$ (for standard elements, i.e., defined by constant functions from) and $\mathtt{Inf}$ (for infinitesimal elements, i.e., elements situated between all positive and negative standard elements in $\mathbb{R}^\omega/D$) has no simple Σ–presentations over $\mathbb{HF}(\mathfrak{M})$.

Remark The structures of kind $\mathbb{R}^\omega/D$ are not always real closed fields. For instance, if D is the Fréchet filter then this structure is not linearly ordered.

Nonpresentability of some nonstandard extensions of $\mathbb{R}$ (independently of concrete constructions)

Theorem

Let $\mathfrak{M}$ be an $\exists$–Steinitz structure of a finite signature and ${}^*\mathbb{R}$ be an arbitrary extension of the ordered field of the reals $\mathbb{R}$ containing infinitesimal elements; and assume that for each function $f : \mathbb{R}^2 \to \mathbb{R}$ there exists a function ${}^*f : ({}^*\mathbb{R})^2 \to {}^*\mathbb{R}$ such that $\langle \mathbb{R}, f \rangle_{f \in \mathcal{F}}$ is an elementary submodel in $\langle {}^*\mathbb{R}, {}^*f \rangle_{f \in \mathcal{F}}$, where $\mathcal{F}$ is the family of all binary functions on $\mathbb{R}$. Then the structure $\langle {}^*\mathbb{R}, \mathtt{St}, \mathtt{Inf} \rangle$ in which the predicate $\mathtt{St}$ distinguishes standard elements from ${}^*\mathbb{R}$ (i.e., elements from $\mathbb{R}$) and $\mathtt{Inf}$ distinguishes infinitesimal elements (i.e., $\mathtt{Inf} = \bigcap_{i<\omega} \left[-\frac{1}{n}, \frac{1}{n}\right]$) has no simple Σ–presentations over $\mathbb{HF}(\mathfrak{M})$.

The idea of the proof

Actually we use decompositions of kind

$$a = \sum_{i<\omega} a_i t^i,$$

where t is an infinitesimal element.
(Details are omitted)

Here we consider Hiblert spaces as structures of kind

$$\mathcal{H} = \langle M, +, \cdot, (\cdot, \cdot) \rangle$$

where $M = H \uplus R$,

• H is the space itself

• R is any set of cardinality 2^ω (the set of reals without any operations and relations)

• $+$ is the predicate defining addition on H

• $\cdot : R \times H \to H$ is the predicate defining the operation of multiplication by scalars

• $(\cdot, \cdot) : H \times H \to R$ is the predicate defining the scalar product

Theorem

Any infinite dimensional Hilbert space has no simple Σ–presentations over $\mathbb{HF}(\mathfrak{M})$, for any $\exists$–Steinitz structure $\mathfrak{M}$.

Such a space is isomorphic to the space ℓ_2 consisting of sequences the reals $(a_i)_{i<\omega}$ satisfying the condition $\sum_{i<\omega} a_i^2 < \infty$ with coordinatewise operations and scalar product
$\left((a)_{i<\omega}, (b)_{i<\omega}\right) = \sum_{i<\omega} a_i b_i$
By this we can assume $H = \ell_2$. Basis:

$$(e_i)(i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Thus, $(a_i)_{i<\omega} = \sum_{i<\omega} a_i e_i$.
$A_i = [0, (i+1)^{-1}] \cdot e_i$, $F_i(x) = (x, e_i)$.
For any sequence of values $F_i(a_i)$, $a_i \in A_i$ take $b = \sum_{i<\omega} (a_i, e_i) e_i$.
We have:

$$F(b) = \left(\sum_{j<\omega} (a_j, e_j) e_j, e_i \right) = (a_i, e_i) = F_i(a_i)$$

Some open questions:

- Does $\mathbb{Q}_p$ have a simple Σ–presentation over $\mathbb{HF}(\mathbb{R})$?
- Does $\mathbb{R}$ have a simple Σ–presentation over $\mathbb{HF}(\mathbb{Q}_p)$?
- Does there exist a nonstandard extension of $\mathbb{R}$ of cardinality 2^ω with a simple Σ–presentation over $\mathbb{HF}(\mathbb{R})$?
- The same questions without the requirement of simplicity of presentations.
- Does any compatible countable theory have a model of the cardinality 2^ω simply presentable over $\mathbb{HF}(\mathbb{R})$? (Without simplicity: yes).

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Thank you !