Generic Muchnik reducibility

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Muchnik reducibility between structures

Definition

If $\mathcal{A}$ and $\mathcal{B}$ are countable structures, then $\mathcal{A}$ is Muchnik reducible to $\mathcal{B}$ (written $\mathcal{A} \leqslant_w \mathcal{B}$) if every ω-copy of $\mathcal{B}$ computes an ω-copy of $\mathcal{A}$.

- $\mathcal{A} \leqslant_w \mathcal{B}$ can be interpreted as saying that $\mathcal{B}$ is intrinsically at least as complicated as $\mathcal{A}$.
- This is a special case of Muchnik reducibility; it might be more precise to say that the problem of presenting the structure $\mathcal{A}$ is Muchnik reducible to the problem of presenting $\mathcal{B}$.
- Muchnik reducibility doesn't apply to uncountable structures.

Various approaches have been used to extend computable structure theory beyond the countable:

- Computability on admissible ordinals (aka α-recursion theory)
- Computability on separable structures, as in computable analysis
- ...

Generic Muchnik reducibility

Noah Schweber extended Muchnik reducibility to arbitrary structures (see Knight, Montalbán, Schweber):

Definition (Schweber)

If $\mathcal{A}$ and $\mathcal{B}$ are (possibly uncountable) structures, then $\mathcal{A}$ is generically Muchnik reducible to $\mathcal{B}$ (written $\mathcal{A} \leqslant_w^* \mathcal{B}$) if $\mathcal{A} \leqslant_w \mathcal{B}$ in some forcing extension of the universe in which $\mathcal{A}$ and $\mathcal{B}$ are countable.

It follows from Shoenfield absoluteness that generic Muchnik reducibility is robust.

Lemma (Schweber)

If $\mathcal{A} \leqslant_w^* \mathcal{B}$, then $\mathcal{A} \leqslant_w \mathcal{B}$ in *every* forcing extension that makes $\mathcal{A}$ and $\mathcal{B}$ countable.

In particular, for countable structures, $\mathcal{A} \leqslant_w^* \mathcal{B} \iff \mathcal{A} \leqslant_w \mathcal{B}$.

Initial example

Definition (Cantor space)

Let $\mathcal{C}$ be the structure with universe 2^ω and predicates $P_n(X)$ that hold if and only if $X(n) = 1$.

Observation (Knight, Montalbán, Schweber)

$$\mathcal{C} \leqslant^*_w (\mathbb{R}, +, \cdot).$$

To understand this example, say that we take a forcing extension that collapses the continuum.

The Turing degrees from the ground model now form a countable ideal I. By absoluteness, this ideal has many of the properties it has in the ground model. It's a jump ideal and much more.

Let $\mathbb{R}_I$ be the reals in I (the ground model's version of $\mathbb{R}$). Similarly, let $\mathcal{C}_I$ denote the restriction of $\mathcal{C}$ to sets in I (the ground model's version of $\mathcal{C}$).

Initial example

Facts

- From a copy of $(\mathbb{R}_I, +, \cdot)$, or even $(\mathbb{R}_I, +, <)$, we can compute an *injective* listing of the sets in I, i.e., one with no repetitions.
- A degree $\mathbf{d}$ computes a copy of $\mathcal{C}_I$ iff it computes an (injective) listing of the sets in I.

This shows that $\mathcal{C}_I \leqslant_w (\mathbb{R}_I, +, <)$. It is even easier to see that $(\mathbb{R}_I, +, <) \leqslant_w (\mathbb{R}_I, +, \cdot)$.

Therefore, $\mathcal{C} \leqslant_w^* (\mathbb{R}, +, <) \leqslant_w^* (\mathbb{R}, +, \cdot)$.

Question (Knight, Montalbán, Schweber)

$$\text{Is } (\mathbb{R}, +, \cdot) \leqslant_w^* \mathcal{C}?$$

No! This was answered by Igusa and Knight, and independently (though later) by Downey, Greenberg, and M.

Facts about $\mathcal{C}$ and $\mathcal{B}$

Definition (Baire space)

Let $\mathcal{B}$ be the structure with universe ω^ω and, for each finite string $\sigma \in \omega^{<\omega}$, a predicate $P_\sigma(f)$ that holds if and only if $\sigma \prec f$.

The following facts were proved by Igusa, Knight; Downey, Greenberg, M.; Igusa, Knight, Schweber; Andrews, Knight, Kuyper, Lempp, M., Soskova.

- $\mathcal{B} \equiv_w^* (\mathbb{R}, +, <) \equiv_w^* (\mathbb{R}, +, \cdot)$. This degree also contains every closed/continuous expansion of $(\mathbb{R}, +, \cdot)$.
- $\mathcal{C} <_w^* \mathcal{B}$.
- $\mathcal{C}' \equiv_w^* \mathcal{B}$.
- The closed/continuous expansions of $\mathcal{C}$ lie in the interval between $\mathcal{C}$ and $\mathcal{B}$.

Question

Is there a generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$?

Definability and post-extension complexity

It is going to be important to understand the complexity of definable sets both before and after the forcing extension.

Definition

We say that a relation R on a structure $\mathcal{M}$ is $\Sigma^c_n(\mathcal{M})$ if it is definable by a computable Σ_n $\mathcal{L}_{\omega_1\omega}$ formula with finitely many parameters.

Theorem (Ash, Knight, Manasse, Slaman; Chisholm)

If $\mathcal{M}$ is countable, then R is $\Sigma^c_n(\mathcal{M})$ if and only if it is relatively intrinsically Σ^0_n, i.e., its image in any ω-copy of $\mathcal{M}$ is Σ^0_n relative to that copy.

Computable objects and satisfaction on a structure are absolute, so:

Corollary

A relation R is $\Sigma^c_n(\mathcal{M})$ if and only if it is relatively intrinsically Σ^0_n in any/every forcing extension that makes $\mathcal{M}$ countable.

Definability and pre-extension complexity

In structures like $\mathcal{C}$ and $\mathcal{B}$, we can also measure the complexity of $\Sigma^c_n(\mathcal{M})$ relations in topological terms.

The calculation depends on the structure:

	Σ^c_2	Σ^c_3	Σ^c_4	Σ^c_5	Σ^c_6	$\dots$
$\mathcal{B}$	$\boldsymbol{\Sigma^1_1}$	$\boldsymbol{\Sigma^1_2}$	$\boldsymbol{\Sigma^1_3}$	$\boldsymbol{\Sigma^1_4}$	$\boldsymbol{\Sigma^1_5}$	$\dots$
$\mathcal{C}$	$\boldsymbol{\Sigma^0_2}$	$\boldsymbol{\Sigma^1_1}$	$\boldsymbol{\Sigma^1_2}$	$\boldsymbol{\Sigma^1_3}$	$\boldsymbol{\Sigma^1_4}$	$\dots$

- These bounds are sharp, e.g., every $\boldsymbol{\Sigma^1_1}$ relation on $\mathcal{B}$ is $\Sigma^c_2(\mathcal{B})$.
- The "lost quantifiers" correspond to the first order quantifiers needed in the normal form for Σ^1_n relations with function/set quantifiers.
- This leads to an easy (and essentially different) separation between the generic Muchnik degrees of $\mathcal{C}$ and $\mathcal{B}$.

Differentiating $\mathcal{C}$ and $\mathcal{B}$ with a linear order

Lemma

There is a linear order $\mathcal{L}$ such that $\mathcal{L} \leqslant_w^* \mathcal{B}$ but $\mathcal{L} \not\leqslant_w^* \mathcal{C}$.

Proof Idea

For $X \subseteq \mathcal{C}$, we define a linear order $\mathcal{L}_X$ that codes X. It is essentially a shuffle sum of delimited ζ-representations of *all* elements of Cantor space along with markers for the sequences not in X.

It is designed so that:

- If X is $\Pi_3^c(\mathcal{B})$, then $\mathcal{L}_X \leqslant_w^* \mathcal{B}$,
- If $\mathcal{L}_X \leqslant_w^* \mathcal{C}$, then X is $\Sigma_4^c(\mathcal{C})$.

Now take $X \subseteq \mathcal{C}$ to be $\mathbf{\Pi_2^1}$ but not $\mathbf{\Sigma_2^1}$. By the analysis on the previous slide:

- X is $\Pi_3^c(\mathcal{B})$, so $\mathcal{L}_X \leqslant_w^* \mathcal{B}$,
- X is not $\Sigma_4^c(\mathcal{C})$, so $\mathcal{L}_X \not\leqslant_w^* \mathcal{C}$. $\square$

A degree strictly between $\mathcal{C}$ and $\mathcal{B}$

Lemma

There is a linear order $\mathcal{L}$ such that $\mathcal{L} \leqslant^*_w \mathcal{B}$ but $\mathcal{L} \nleqslant^*_w \mathcal{C}$.

But linear orders are bad at coding:

Lemma

If $\mathcal{L}$ is a linear order, then $\mathcal{B} \nleqslant^*_w \mathcal{C} \sqcup \mathcal{L}$.

Following the Downey, Greenberg, M. proof that $\mathcal{B} \nleqslant^*_w \mathcal{C}$, we show that a generic countable presentation of $\mathcal{C} \sqcup \mathcal{L}$ does not compute a copy of $\mathcal{B}$. The key fact used about linear orders is that their $\sim_2$-equivalence classes are tame (Knight 1986).

Now let $\mathcal{M} = \mathcal{C} \sqcup \mathcal{L}$, where $\mathcal{L}$ is the linear order from the first lemma.

Corollary

There is a structure $\mathcal{M}$ such that $\mathcal{C} <^*_w \mathcal{M} <^*_w \mathcal{B}$.

Great! But…not the most satisfying example.

What kind of example would we like?

The initial attempts to find an intermediate degree involved natural expansions of $\mathcal{C}$, but without success. For example:

- $(\mathcal{C}, \oplus) \equiv^*_w (\mathcal{C}, \sigma) \equiv^*_w \mathcal{B}$, where σ is the shift operator on 2^ω.
- $(\mathcal{C}, \subseteq) \equiv^*_w (\mathcal{C}, \triangle) \equiv^*_w \mathcal{C}$.

Another approach would be to expand $\mathcal{C}$ with sufficiently generic relations. Greenberg, Igusa, Turetsky, and Westrick tried a version of this that involved adding infinitely many unary relations.

In both cases, we considered *expansions* of $\mathcal{C}$.

Open Question

Is there an expansion of $\mathcal{C}$ that is strictly between $\mathcal{C}$ and $\mathcal{B}$?

Expansions of $\mathcal{C}$ above $\mathcal{B}$

Let $\mathcal{M} = (\mathcal{C}, \text{Stuff})$ be an expansion of $\mathcal{C}$. First, we want a criterion that guarantees that $\mathcal{M} \geqslant_w^* \mathcal{B}$.

- If the set $\mathcal{F} \subset 2^\omega$ of sequences with finitely many ones is $\Delta_1^c(\mathcal{M})$, i.e., computable in every ω-copy of $\mathcal{M}$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.
  - Why? There is a natural bijection between $\mathcal{B}$ and $\mathcal{C} \setminus \mathcal{F}$.

- If $\mathcal{F}$ is $\Delta_2^c(\mathcal{M})$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.
  - Add a little injury.
  - This is how we show, for example, that $(\mathcal{C}, \oplus) \geqslant_w^* \mathcal{B}$.

- If any countable dense set is $\Delta_2^c(\mathcal{M})$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.

- If there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ with a countable dense $\mathcal{Q} \subset \mathcal{P}$ that is $\Delta_2^c(\mathcal{M})$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.

Expansions of $\mathcal{C}$ above $\mathcal{B}$

- If there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ with a countable dense $\mathcal{Q} \subset \mathcal{P}$ that is $\Delta_2^c(\mathcal{M})$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.

Lemma

If $\mathcal{M} \leqslant_w^* \mathcal{B}$ and $R \subseteq \mathcal{C}$ is $\Delta_2^c(\mathcal{M})$, then it is $\Delta_2^c(\mathcal{B})$, i.e., Borel.

Lemma (Hurewicz)

If $R \subseteq \mathcal{C}$ is Borel but not $\boldsymbol{\Delta_2^0}$, then there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ such that either $\mathcal{P} \cap R$ or $\mathcal{P} \setminus R$ is countable and dense in $\mathcal{P}$.

Putting it all together (and noting that arity doesn't matter):

Lemma

If $\mathcal{M} \leqslant_w^* \mathcal{B}$ is an expansion of $\mathcal{C}$ and $R \subseteq \mathcal{C}^n$ is $\Delta_2^c(\mathcal{M})$ but not $\boldsymbol{\Delta_2^0}$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.

Tameness and dichotomy

In the contrapositive (and using the fact that $\mathbf{\Delta^0_2} = \Delta^c_2(\mathcal{C})$):

Tameness Lemma

If $\mathcal{M} <^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$, then $\Delta^c_2(\mathcal{M}) = \Delta^c_2(\mathcal{C})$.

Dichotomy Theorem for Closed Expansions

If $\mathcal{M} \leqslant^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$ by closed relations (and/or continuous functions), then either $\mathcal{M} \equiv^*_w \mathcal{C}$ or $\mathcal{M} \equiv^*_w \mathcal{B}$.

Proof Idea

For a tuple $\overline{X} \subset \mathcal{C}$, let $p(\overline{X})$ be the (code for the) complete positive $\Sigma_1(\mathcal{M})$ type of $\overline{X}$. The relation that holds only on tuples of the form $(\overline{X}, p(\overline{X}))$ is $\Delta^c_2(\mathcal{M})$.

If it is not $\Delta^c_2(\mathcal{C})$, then $\mathcal{M} \geqslant^*_w \mathcal{B}$.

If it is $\Delta^c_2(\mathcal{C})$, then a delicate injury argument can be used to prove that $\mathcal{M} \leqslant^*_w \mathcal{C}$. $\square$

Another dichotomy result

Combined with work of Greenberg, Igusa, Turetsky, and Westrick:

Dichotomy Theorem for Unary Expansions

If $\mathcal{M} \leqslant_w^* \mathcal{B}$ is an expansion of $\mathcal{C}$ by countably many unary relations, then either $\mathcal{M} \equiv_w^* \mathcal{C}$ or $\mathcal{M} \equiv_w^* \mathcal{B}$.

- If $\mathcal{M}$ is an expansion of $\mathcal{C}$ by finitely many $\boldsymbol{\Delta_2^0}$ unary relations, then $\mathcal{M} \leqslant_w^* \mathcal{C}$. This is a fairly simple finite injury argument.
- Expansions by infinitely many closed unary relations need not be below $\mathcal{C}$: For $\sigma \in 2^{<\omega}$, let U_σ hold only on $\sigma 0^\omega$. Then the set of sequences with finitely many ones is $\Sigma_1^c(\mathcal{C}, \{U_\sigma\}_{\sigma \in 2^{<\omega}})$.
- Greenberg, et al. supplied the right condition distinguishing the cases, and one direction of the proof.

The dichotomy results kill off a lot of possible natural (and many unnatural) examples of expansions.

Recent progress

Based partly on conversations with Turetsky and Gura, I am pretty sure the following is true.

Using Marker extensions, we can get structures with the following "complexity profiles":

	Σ^c_2	Σ^c_3	Σ^c_4	Σ^c_5	Σ^c_6	$\dots$
$\mathcal{B}$	$\mathbf{\Sigma^1_1}$	$\mathbf{\Sigma^1_2}$	$\mathbf{\Sigma^1_3}$	$\mathbf{\Sigma^1_4}$	$\mathbf{\Sigma^1_5}$	$\dots$
$\mathcal{M}_1$	$\mathbf{\Sigma^0_2}$	$\mathbf{\Sigma^1_2}$	$\mathbf{\Sigma^1_3}$	$\mathbf{\Sigma^1_4}$	$\mathbf{\Sigma^1_5}$	$\dots$
$\mathcal{M}_2$	$\mathbf{\Sigma^0_2}$	$\mathbf{\Sigma^1_1}$	$\mathbf{\Sigma^1_3}$	$\mathbf{\Sigma^1_4}$	$\mathbf{\Sigma^1_5}$	$\dots$
$\mathcal{M}_3$	$\mathbf{\Sigma^0_2}$	$\mathbf{\Sigma^1_1}$	$\mathbf{\Sigma^1_2}$	$\mathbf{\Sigma^1_4}$	$\mathbf{\Sigma^1_5}$	$\dots$
			$\vdots$			
$\mathcal{C}$	$\mathbf{\Sigma^0_2}$	$\mathbf{\Sigma^1_1}$	$\mathbf{\Sigma^1_2}$	$\mathbf{\Sigma^1_3}$	$\mathbf{\Sigma^1_4}$	$\dots$

- Again, these bounds are sharp.
- $\mathcal{C} <^*_w \cdots <^*_w \mathcal{M}_3 <^*_w \mathcal{M}_2 <^*_w \mathcal{M}_1 <^*_w \mathcal{B}$.

Open questions

1. Can an expansion of $\mathcal{C}$ be strictly between $\mathcal{C}$ and $\mathcal{B}$? (In particular, the non-unary $\mathbf{\Delta^0_2}$ case is open.)
2. Are the degrees of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$ the only degrees strictly between $\mathcal{C}$ and $\mathcal{B}$?
3. Are there incomparable degrees between $\mathcal{C}$ and $\mathcal{B}$?

These questions are related. For example:

Fact. Any Borel expansion of $\mathcal{C}$ that is not above $\mathcal{B}$ has the same complexity profile as $\mathcal{C}$. So a positive answer to 1 gives a negative answer to 2.

We have focused on $\mathcal{C}$ and $\mathcal{B}$ (and a couple of other degrees). What else are generic Muchnik degrees good for?

Thank you.

And thanks to Rod
for being a great friend and mentor!