TORSION-FREE ABELIAN GROUPS WITH OPTIMAL SCOTT FAMILIES

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- My first-ever small article had 3 results, all proved and published by Rod some 10 years before me.
- Three theorems on constructive abelian groups, Notices of NSU, 2007, **rejected**.)
- In 2007 Julia sent me a preprint by Rod and Antonio where they proved the isomorphism problem for TFAG is Σ^1_1-complete.
- I had my revenge at the end of 2007. Using a **rather simple** argument I solved a problem of Rod on jump degrees of TFAGs.

- Who is this MAN who keeps proving MY THEOREMS?

- Our productive collaboration began in 2010 when I came to visit Wellington.

The result that I want to discuss is **not** joint with Rod, but it is **technically related** to our joint work.

In the 1980's Ventsov introduced **relative computable categoricity** which is stronger than computable categoricity but is **fully relativisable**.

Ash defined **relative Δ^0_α categoricity**.

It is well-known that a structure is relatively Δ^0_α-categorical iff the orbits of its initial segments can be uniformly described by Σ^c_α-formulae.

Computable categoricity has been described in many standard classes, but (even relative) Δ^0_α-categoricity in a natural class is usually hard to explicitly describe when α gets large.

We rather seek for **optimal examples**, i.e. structures that are Δ^0_α-categorical but not Δ^0_β-categorical for $\beta < \alpha$.

Theorem

In each of the listed below classes, we indicate the form of computable ordinals $\alpha > 1$ for which optimal examples of relative Δ^0_α-categoricity exist in the class. In each case δ is either 0 or a limit ordinal, and $k \in \omega$.

1. Linear orders, each α of the form $\delta + 2k$ (Ash).
2. Boolean algebras, each α of the form $\delta + 2k + 2$ or δ (Knight).
3. Abelian p-groups, any α (Barker).
4. Ordered abelian groups, each α of the form $1 + \delta + 2k$ (M.).
5. Real closed fields, each α of the form $1 + \delta + 2k$ (Ocasio-Gonzalez).
6. Completely decomposable groups and any $\alpha \leq 5$ (Downey and M.).
7. Effectively universal classes including graphs, 2-step nilpotent groups, and for any computable α (folklore).

Why do we care?

Such results usually provide us with:

- Lots of non-trivial but well-understood examples of structures in the class.
- A detailed analysis of a reach enough subclass of the class (e.g., superatomic BAs for the class of BAs).
- New advanced computability-theoretic techniques (such as Ash's α-systems or novel strategies).
- New algebraic techniques (such as P-independence).
- $\mathcal{L}^c_{\omega_1\omega}$-definability analysis.

All these tools typically find applications not necessarily related to categoricity.

Note torsion-free abelian groups (TFAG) are not listed in the “big theorem” for $\alpha > 5$. I claim this is **very surprising.** (Why?)

In the 1960’s, Maltsev initiated the systematic study of computable algebras in the USSR.

TFAG was one of Maltsev’s favourite algebraic classes. This has roots in his mathematical childhood.

His deep interest in the subject stimulated the development of the field within the former USSR.

TFAG is not listed in the “big theorem” for $\alpha > 5$. I claim this is **NOT very surprising.** (Why?)

All results on computable TFAGs can be divided into 3 main groups:

- Local theorems.
- Anti-structure theorems.
- Results on completely decomposable groups.

An example of a **local theorem**:

Theorem (Essentially Nurtazin for TFAG, Dobrica for arbitrary AG)

Every computable TFAG has a computable presentation with a computable Prüfer basis.

There is nothing special about TFAG here:

Theorem (Harrison-Trainor, M., Montalban 2015)

The same is true about many other classes with r.i.c.e. pregeometries, including:

- Real-closed fields and algebraic independence,
- Differentially-closed fields and δ-independence,
- Difference-closed fields and transformal independence,
- Archimedean ordered abelian groups and linear independence [Goncharov, Lempp, Solomon]
- Existencially closed valued fields [Harrison-Trainor 2016]

Examples of an **anti-structure theorems**:

Theorem (Downey and Montalbán 2008)

The isomorphism problem

$$\{(i,j) : A_i \cong A_j \text{ and } A_i, A_j \in TFAG\}$$

for TFAG is Σ^1_1-complete.

Theorem (Riggs 2015)

The decomposition problem

$$\{i : A_i \text{ splits into a non-trivial direct sum and } A_i \in TFAG\}$$

for TFAG is Σ^1_1-complete.

These results provide us with a sequence of examples of **very bad TFAGs.** We illustrate the groups are **sufficiently bad** to do the job, but we do not fully understand these examples (algebraically).

A group A is **completely decomposable** if it splits into a direct sum of subgroups of the rationals $(\mathbb{Q}, +)$:

$$A = \bigoplus_i A_i, \quad \text{with each } A_i \leq \mathbb{Q}$$

Examples of results on completely decomposable groups:

Theorem (Downey and M., 2013)

The index set of completely decomposable groups is Σ^0_7.

Theorem (Downey and M., 2013)

Every computable completely decomposable group is relatively Δ^0_5-categorical, and $\alpha = 5$ is optimal.

None of these results provide us with “arbitrarily complex” but well-understood examples of constructive TFAGs.

(Optimal examples of relatively Δ^0_α-categorical structures must be of this sort.)

Question (Goncharov, around 2007)

Provide optimal examples of (relatively) Δ^0_α-categorical torsion-free abelian groups for arbitrarily large computable α.

1. **We know CDG fails to do the job.** (But this would be the most natural subclass of TFAG.)
2. However, proving that it fails the task required the development of the machinery of *P*-independence, see:
   - Computable Completely Decomposable Groups, Downey and M., TAMS 2014.
   - Effectively Categorical Abelian Groups, Downey and M., J. of Alg. 2013.

1. **Group-eplags** (Hjorth, based on folklore from abelian group theory) should be the next-most natural subclass of TFAGs.
2. Group-eplags are TFAGs that kind of look like graphs.
3. There are several technical problems related to $\mathcal{L}^{c}_{\omega_1\omega}$-definability.
4. Nonetheless, even partial $\mathcal{L}^{c}_{\omega_1\omega}$-definability analysis was useful:
   - Jump Degrees of Torsion-Free Abelian Groups, JSL 2012, with Andersen, Kach, and Solomon.
   - New Degree Spectra of Abelian Groups, NDJFL (to appear).

One of the main obstacles was the (seemingly necessary) use of finite divisibility conditions.

Finite divisibility conditions lead to many pathologies in TFAG, such as anomalous decompositions etc. Eliminating some of the effects using quasi-isomorphism smoothens the situation.

There was no machinery beyond P-independence.

One would hope that some textbook or paper on abelian groups could help, but no.

- Ideally, we would like our examples to intuitively resemble something that we already know (such as superatomic BAs or reduced abelian p-groups etc.)
- In other words, we want some natural notion of “depth”, or something similar to the Cantor-Bendixon derivative in our subclass of TFAG.

The techniques used for abelian p-groups are far less sophisticated and are almost totally different from those used in TFAG.

Theorem (M. 2016)

For every computable even successor ordinal α there exists a torsion-free abelian group $\mathcal{G}_\alpha$ which is relatively Δ^0_α-categorical, but not $\Delta^0_{\alpha-1}$-categorical.

- The theorem is witnessed by the simplest group-eplags one could hope for.
- We use a new sort of **jump inversion** to **eliminate finite divisibility**.
- We introduce a **new algebraic technique** that resembles **stripping** (this is related to our recent work with Rod and Selwyn on abelian *p*-groups).
- An even more refined $\mathcal{L}_{\omega_1\omega}$-definability analysis.
- A lucky application of *P*-independence.

Some of these ideas have already led to nice bi-product results. I think it is not the end.

Thanks!