Multiple Genericity

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Recall...

Let S be a set of strings. We say that a set $A \subseteq \omega$...

- meets S if there is some $\sigma \prec A$ with $\sigma \in S$
- avoids S if there is some $\sigma \prec A$ such that $(\forall \tau \succcurlyeq \sigma)(\tau \notin S)$.

We say that A is *n*-generic if it meets or avoids every Σ^0_n set of strings.

A set of string S is dense if for every string $\tau \in 2^{<\omega}$, there is some $\sigma \succcurlyeq \tau$ with $\sigma \in S$.
We say that A is weakly *n*-generic if it meets every dense Σ^0_n set of strings.

We have

$$n\text{-generic} \implies \text{weakly } n\text{-generic} \implies (n-1)\text{-generic} \implies \dots$$
$$\implies \text{weakly 2-generic} \implies \text{1-generic} \implies \text{weakly 1-generic.}$$

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What about meeting or avoiding 2-c.e. sets of strings?
This is equivalent to 2-genericity.

Extension functions

Definition

An extension function is a partial function $f : 2^{<\omega} \to 2^{<\omega}$ such that

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1-generic $\equiv$ meet or avoid the range of every partial computable extension function.

weakly 2-generic $\equiv$ meet the range of every total $\emptyset'$-computable extension function.

Pb-genericity

Pb-genericity

Consider a total extension function f for which there is

- a total computable function $\hat{f} : 2^{<\omega} \times \omega \to 2^{<\omega}$, and
- a primitive recursive function $h : 2^{<\omega} \to \omega$

such that

1. $\lim_s \hat{f}(\sigma, s) = f(\sigma)$,
2. $|\{s \in \omega : \hat{f}(\sigma, s+1) \neq \hat{f}(\sigma, s)\}| < h(\sigma)$.

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Here the bound is given in advance.

Definition (Downey, Jockusch, and Stob)

A degree **a** is array computable if

$$(\exists f \ \omega\text{-c.a.})(\forall g \leqslant_{\mathrm{T}} \mathbf{a})(f \text{ dominates } g).$$

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Theorem (Downey, Jockusch, and Stob)

Every pb-generic is of array noncomputable degree, and every array noncomputable degree computes a pb-generic.

What if the bound is not given in advance?

Replace the primitive recursive function h from before with a total computable function.

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Replace the primitive recursive function h from before with a total computable function.

Definition

A set is weakly ω-change generic if it meets the range of every total ω-c.a. extension function.

These were called c-generics by Schaeffer.

Definition (DJS; Downey and Greenberg)

A degree $\mathbf{a}$ is array computable (or uniformly ω-c.a. dominated) if

$$(\exists f\ \omega\text{-c.a.})(\forall g \leqslant_T \mathbf{a})(\,f \text{ dominates } g\,).$$

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Theorem

Every weakly ω-change generic is of not ω-c.a. dominated degree, and every not ω-c.a. dominated degree computes a weakly ω-change generic.

What about sets of strings which are not dense?

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Definition

An ω-change test is a sequence of pairs $\langle f_s, o_s \rangle_{s<\omega}$ of uniformly computable functions $f_s : 2^{<\omega} \to 2^{<\omega}$ and $o_s : 2^{<\omega} \to \omega + 1$ such that for all $\sigma \in 2^{<\omega}$ and $s \in \omega$,

- $f_s(\sigma) \succcurlyeq \sigma$,
- $o_{s+1}(\sigma) \leqslant o_s(\sigma)$, and
- if $f_{s+1}(\sigma) \neq f_s(\sigma)$, then $o_{s+1}(\sigma) < o_s(\sigma)$.

We let the range of the test $\langle f_s, o_s \rangle_{s<\omega}$ be

$$\{\lim_s f_s(\sigma) : \lim_s o_s(\sigma) < \infty\}.$$

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Definition

A set is ω-change generic if it meets or avoids the range of every ω-change test.

What about ordinals other than just ω?

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We can write any ordinal α uniquely in Cantor Normal Form:

$$\omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \ldots + \omega^{\alpha_k} n_k$$

where $n_i \in \omega$ are nonzero and $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ are ordinals.

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If we have a computable well-ordering $\mathcal{R}$, we say that it is canonical if, roughly, we can computably go from any element $z \in \mathcal{R}$, to the Cantor Normal Form of the ordinal it represents.

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If we have a computable well-ordering $\mathcal{R}$, we say that it is canonical if, roughly, we can computably go from any element $z \in \mathcal{R}$, to the Cantor Normal Form of the ordinal it represents.

We have

$$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}.$$

For every $\alpha \leqslant \varepsilon_0$, there is a computable canonical well-ordering with order-type α.

For any $\alpha \leqslant \varepsilon_0$, we can define an α-change test exactly as before, just replacing ω with α.

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We say an α-change test is total if for all σ, $\lim_s o_s(\sigma) < \infty$.

An α-order function is a total computable function $h : 2^{<\omega} \to \alpha$ that is non-decreasing (w.r.t. the length of the strings, say) and with range unbounded in α.

We say an α-change test is h-bounded if for all σ, $o_0(\sigma) < h(\sigma)$.

We say that a degree **a** is...

- α-change generic if there is a set $A \in \mathbf{a}$ which meets or avoids the range of every α-change test,
- weakly α-change generic if there is a set $A \in \mathbf{a}$ which meets the range of every total α-change test, and
- uniformly α-change generic if for some (or equivalently, all) α-order functions h, there is a set $A \in \mathbf{a}$ which meets the range of every h-bounded α-change test.

weakly 2-generic

$\Downarrow$

$\vdots$

$\Downarrow$

ω^2-change generic

$\Downarrow$

weakly ω^2-change generic

$\Downarrow$

uniformly ω^2-change generic

$\Downarrow$

ω-change generic

$\Downarrow$

weakly ω-change generic

$\Downarrow$

uniformly ω-change generic $\equiv$ pb-generic

$\Downarrow$

1-change generic $\equiv$ 1-generic

ω-c.a. functions $\rightsquigarrow$ α-c.a. functions

Definition (Downey and Greenberg)

A degree **a** is uniformly α-c.a. dominated if

$$(\exists f \ \alpha\text{-c.a.})(\forall g \leqslant_T \mathbf{a})(\ f \text{ dominates } g\ ).$$

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Theorem (Downey and Greenberg)
For any α that is a power of ω, there is a degree (in fact, a c.e. degree) that is α-c.a. dominated, but not uniformly α-c.a. dominated.

Using this result, we can get a degree which is uniformly α-change generic, but not weakly α-change generic.

Luckily, computing α-change generics is not so straightforward.

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Theorem

Every not uniformly $\omega^{\beta+1}$-c.a. dominated degree computes computes an ω^{β}-change generic degree.

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Every not ω^{β}-c.a. dominated c.e. degree computes an ω^{β}-change generic degree.

Can we get rid of the c.e. assumption here?

No!

Theorem

For any α that is a power of ω, there is a Δ^0_2 not α-c.a. dominated degree which does not compute an α-change generic degree.

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Theorem

For any α that is a power of ω, there is a Δ^0_2 not α-c.a. dominated degree which does not compute an α-change generic degree.

This gives us a degree which is weakly α-change generic but not α-change generic.

weakly 2-generic

$\Downarrow$

$\vdots$

$\Downarrow$

ω^2-change generic

$\Downarrow$

weakly ω^2-change generic

$\Downarrow$

uniformly ω^2-change generic

$\Downarrow$

ω-change generic

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weakly ω-change generic

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uniformly ω-change generic $\equiv$ pb-generic

$\Downarrow$

1-change generic $\equiv$ 1-generic

Downward density

Definition

We say that a class of degrees $\mathcal{D}$ is *downwards dense* below a degree $\mathbf{a}$ if for every noncomputable degree $\mathbf{b} \leqslant_T \mathbf{a}$, there is some $\mathbf{c} \in \mathcal{D}$ with $\mathbf{c} \leqslant_T \mathbf{b}$.

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Theorem (Martin)

For all $n \geqslant 2$, the n-generic degrees are downwards dense below any n-generic degree.

Theorem (Chong and Downey)

There is a 1-generic below $\emptyset''$ *which bounds a minimal degree.*

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But below $\emptyset'$,...

Theorem (Chong and Jockusch)

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Theorem (Chong and Jockusch)

The 1-generic degrees are downwards dense below 1-generic degrees below $\emptyset'$.

Theorem (Schaeffer)

Downward density fails for pb-generics below $\emptyset'$.

Theorem

The α-change generic degrees are downwards dense below α-change generic degrees below $\emptyset'$.

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Theorem

There is a Δ^0_2 weakly α-change generic degree which computes a noncomputable degree that does not compute a uniformly α-change generic degree. So downward density fails for weakly α-change and uniformly α-change generic degrees.

From this we can get a degree which is ω^β-change generic, but not uniformly $\omega^{\beta+1}$-change generic.

Further directions

What are these generics good for? (Typical behaviour)

Interactions with randomness? (Computably bounded randoms)

Lowness for these notions?

Thanks