

Multiple Genericity

Michael McInerney, Nanyang Technological University
Joint work with Selwyn Ng, also at NTU

Computability and Complexity Symposium 2017

January 8, 2017

Recall...

Let S be a set of strings. We say that a set $A \subseteq \omega \dots$

- ▶ **meets** S if there is some $\sigma \prec A$ with $\sigma \in S$
- ▶ **avoids** S if there is some $\sigma \prec A$ such that $(\forall \tau \succ \sigma)(\tau \notin S)$.

We say that A is **n -generic** if it meets or avoids every Σ_n^0 set of strings.

A set of string S is **dense** if for every string $\tau \in 2^{<\omega}$, there is some $\sigma \succ \tau$ with $\sigma \in S$.

We say that A is **weakly n -generic** if it meets every dense Σ_n^0 set of strings.

We have

$$\begin{aligned} n\text{-generic} &\implies \text{weakly } n\text{-generic} \implies (n-1)\text{-generic} \implies \dots \\ &\implies \text{weakly } 2\text{-generic} \implies 1\text{-generic} \implies \text{weakly } 1\text{-generic}. \end{aligned}$$

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What about meeting or avoiding 2-c.e. sets of strings?
This is equivalent to 2-genericity.

Extension functions

Definition

An **extension function** is a partial function $f : 2^{<\omega} \rightarrow 2^{<\omega}$ such that

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1-generic \equiv meet or avoid the range of every partial computable extension function.

weakly 2-generic \equiv meet the range of every total \emptyset' -computable extension function.

Pb-genericity

Pb-genericity

Consider a total extension function f for which there is

- ▶ a total computable function $\hat{f} : 2^{<\omega} \times \omega \rightarrow 2^{<\omega}$, and
- ▶ a **primitive recursive** function $h : 2^{<\omega} \rightarrow \omega$

such that

1. $\lim_s \hat{f}(\sigma, s) = f(\sigma)$,
2. $|\{s \in \omega : \hat{f}(\sigma, s+1) \neq \hat{f}(\sigma, s)\}| < h(\sigma)$.

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Here the bound is given **in advance**.

Definition (Downey, Jockusch, and Stob)

A degree \mathbf{a} is **array computable** if

$$(\exists f \text{ } \omega\text{-c.a.})(\forall g \leq_{\mathbf{T}} \mathbf{a})(f \text{ dominates } g).$$

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Theorem (Downey, Jockusch, and Stob)

*Every pb-generic is of array **non**computable degree, and every array noncomputable degree computes a pb-generic.*

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Replace the primitive recursive function h from before with a total computable function.

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Replace the primitive recursive function h from before with a total computable function.

Definition

A set is **weakly ω -change generic** if it meets the range of every total ω -c.a. extension function.

These were called c -generics by Schaeffer.

Definition (DJS; Downey and Greenberg)

A degree \mathbf{a} is **array computable** (or **uniformly ω -c.a. dominated**) if

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Theorem

Every weakly ω -change generic is of not ω -c.a. dominated degree, and every not ω -c.a. dominated degree computes a weakly ω -change generic.

What about sets of strings which are not dense?

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Definition

An ω -change test is a sequence of pairs $\langle f_s, o_s \rangle_{s < \omega}$ of uniformly computable functions $f_s : 2^{<\omega} \rightarrow 2^{<\omega}$ and $o_s : 2^{<\omega} \rightarrow \omega + 1$ such that for all $\sigma \in 2^{<\omega}$ and $s \in \omega$,

- ▶ $f_s(\sigma) \succcurlyeq \sigma$,
- ▶ $o_{s+1}(\sigma) \leq o_s(\sigma)$, and
- ▶ if $f_{s+1}(\sigma) \neq f_s(\sigma)$, then $o_{s+1}(\sigma) < o_s(\sigma)$.

We let the range of the test $\langle f_s, o_s \rangle_{s < \omega}$ be

$$\left\{ \lim_s f_s(\sigma) : \lim_s o_s(\sigma) < \infty \right\}.$$

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Definition

A set is ω -change generic if it meets or avoids the range of every ω -change test.

What about ordinals other than just ω ?

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We can write any ordinal α uniquely in **Cantor Normal Form**:

$$\omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \dots + \omega^{\alpha_k} n_k$$

where $n_i \in \omega$ are nonzero and $\alpha_1 > \alpha_2 > \dots > \alpha_k$ are ordinals.

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If we have a computable well-ordering \mathcal{R} , we say that it is **canonical** if, roughly, we can computably go from any element $z \in \mathcal{R}$, to the Cantor Normal Form of the ordinal it represents.

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We have

$$\varepsilon_0 = \sup \{ \omega, \omega^\omega, \omega^{\omega^\omega}, \dots \}.$$

For every $\alpha \leq \varepsilon_0$, there is a computable canonical well-ordering with order-type α .

For any $\alpha \leq \varepsilon_0$, we can define an α -change test exactly as before, just replacing ω with α .

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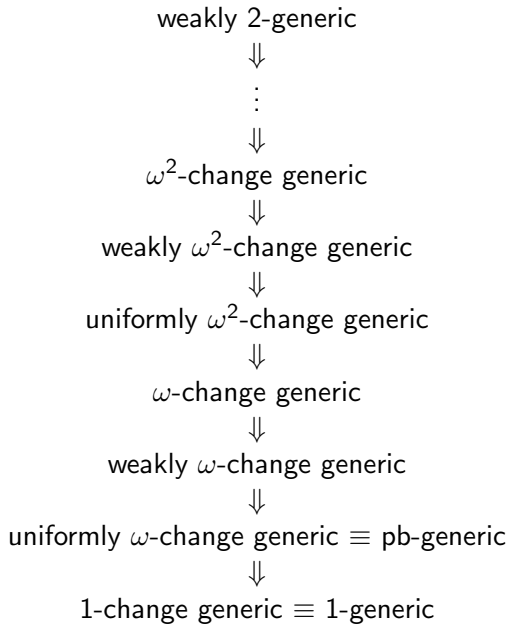
We say an α -change test is **total** if for all σ , $\lim_s o_s(\sigma) < \infty$.

An **α -order function** is a total computable function $h : 2^{<\omega} \rightarrow \alpha$ that is non-decreasing (w.r.t. the length of the strings, say) and with range unbounded in α .

We say an α -change test is **h -bounded** if for all σ , $o_0(\sigma) < h(\sigma)$.

We say that a degree \mathbf{a} is...

- ▶ α -change generic if there is a set $A \in \mathbf{a}$ which meets or avoids the range of every α -change test,
- ▶ **weakly** α -change generic if there is a set $A \in \mathbf{a}$ which meets the range of every **total** α -change test, and
- ▶ **uniformly** α -change generic if for some (or equivalently, all) α -order functions h , there is a set $A \in \mathbf{a}$ which meets the range of every **h -bounded** α -change test.



ω -c.a. functions \rightsquigarrow α -c.a. functions

Definition (Downey and Greenberg)

A degree \mathbf{a} is **uniformly α -c.a. dominated** if

$$(\exists f \text{ } \alpha\text{-c.a.})(\forall g \leq_T \mathbf{a})(f \text{ dominates } g).$$

A degree \mathbf{a} is **α -c.a. dominated** if

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Theorem (Downey and Greenberg)

For any α that is a power of ω , there is a degree (in fact, a c.e. degree) that is α -c.a. dominated, but not uniformly α -c.a. dominated.

Using this result, we can get a degree which is uniformly α -change generic, but not weakly α -change generic.

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Can we get rid of the c.e. assumption here?

No!

Theorem

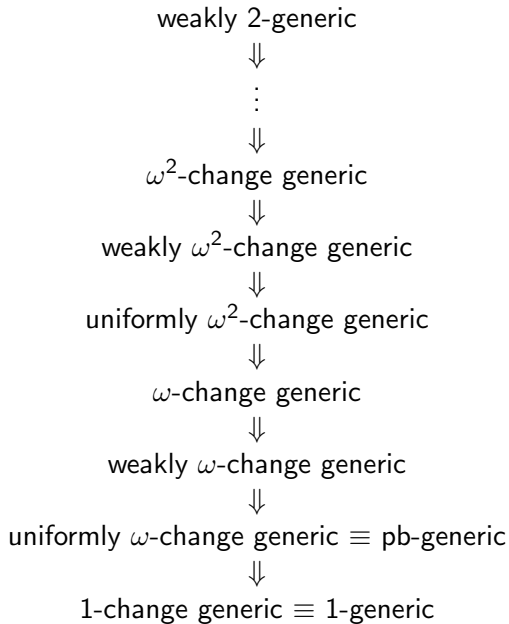
For any α that is a power of ω , there is a Δ_2^0 not α -c.a. dominated degree which does not compute an α -change generic degree.

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Theorem

For any α that is a power of ω , there is a Δ_2^0 not α -c.a. dominated degree which does not compute an α -change generic degree.

This gives us a degree which is weakly α -change generic but not α -change generic.



Downward density

Definition

We say that a class of degrees \mathcal{D} is *downwards dense* below a degree \mathbf{a} if for every noncomputable degree $\mathbf{b} \leq_T \mathbf{a}$, there is some $\mathbf{c} \in \mathcal{D}$ with $\mathbf{c} \leq_T \mathbf{b}$.

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Theorem (Martin)

For all $n \geq 2$, the n -generic degrees are downwards dense below any n -generic degree.

Theorem (Chong and Downey)

There is a 1-generic below \emptyset'' which bounds a minimal degree.

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But below \emptyset' ,...

Theorem (Chong and Jockusch)

The 1-generic degrees are downwards dense below 1-generic degrees below \emptyset' .

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But below \emptyset' ,...

Theorem (Chong and Jockusch)

The 1-generic degrees are downwards dense below 1-generic degrees below \emptyset' .

Theorem (Schaeffer)

Downward density fails for pb-generics below \emptyset' .

Theorem

The α -change generic degrees are downwards dense below α -change generic degrees below \emptyset' .

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Theorem

There is a Δ_2^0 weakly α -change generic degree which computes a noncomputable degree that does not compute a uniformly α -change generic degree. So downward density fails for weakly α -change and uniformly α -change generic degrees.

From this we can get a degree which is ω^β -change generic, but not uniformly $\omega^{\beta+1}$ -change generic.

Further directions

What are these generics good for? (Typical behaviour)

Interactions with randomness? (Computationally bounded randomness)

Lowness for these notions?

Thanks