Tutorial Series on Reverse Mathematics

Denis R. Hirschfeldt — University of Chicago

2017 NZMRI Summer School, Napier, New Zealand

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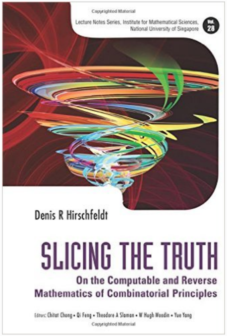

Part I: Background

A Bit of Historical Context

A Bit of Historical Context: Abstraction and the Loss of Certainty

Concrete, algorithmic mathematics

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We can also compare theorems in terms of implication over B.

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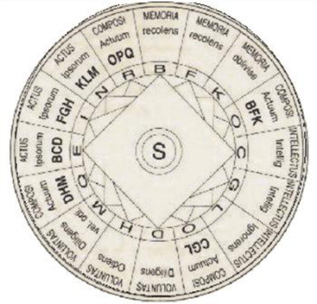
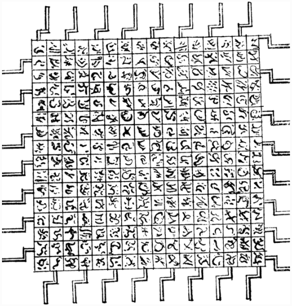
A Llullian circle



The writing machine at the Grand Academy of Lagado (*Gulliver's Travels*, 1726)



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Church-Turing Thesis: This definition captures the intuitive notion of "computable".

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Computability theory has tools to compare such objects.

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Then so is $f(e) = \begin{cases} \Phi_e(e) + 1 & \text{if } \langle e, e\rangle \in \emptyset' \\ 0 & \text{otherwise.} \end{cases}$

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A similar proof shows that there is no effective list of all total computable functions.

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The degree of the join $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$ is the least upper bound of the degrees of A and B.

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Thm (Friedberg; Muchnik). There are noncomputable, incomplete c.e. sets.

There are also non-c.e. sets that are computable relative to $\emptyset'$, including co-c.e. sets but also many others.

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E.g., X' is not computable relative to X, and is complete for sets c.e. relative to X.

Part II: Computability-Theoretic Comparison

An Example: Versions of König's Lemma

Trees and Paths

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Put a topology on $\mathbb{N}^{\omega}$ by taking $\{X : \sigma \prec X\}$ as basic open sets.

Then $\mathcal{C}$ is closed iff it is the set of paths on a tree.

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Put a measure on 2^{ω} by letting $\mu(\{X : \sigma \prec X\}) = 2^{-|\sigma|}$.

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Bounded König's Lemma: Every infinite binary tree T s.t.

$$|\{\sigma \in T : |\sigma| = n\}| < c$$

for some c has a path.

Versions of König's Lemma

KL: Infinite, finitely branching trees have paths.

WKL: Infinite binary trees have paths.

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WKL: Find an element of a closed set.

WWKL: Find an element of a closed of positive measure.

BKL: Find an element of a finite set.

Bounded König's Lemma and Computability

Let T be a computable infinite binary tree s.t.
$|\{\sigma \in T : |\sigma| = n\}| < c$ for all n.

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There is a $\sigma \in T$ extended by a unique path P on T.

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Thus BKL is computably true.

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This theorem relativizes: If the binary tree T is computable relative to X then T has a path P s.t. $(P \oplus X)' \leqslant_{\mathrm{T}} X'$.

Computable Entailment

Second-Order Statements

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We can encode finite objects as natural numbers: e.g., strings, rationals, finite sets, ...

We can encode countably infinite objects as sets of natural numbers: e.g., infinite sequences, trees, groups, reals, ...

So we might encode a $\sigma \in 2^{<\omega}$ of length n as $2\sigma(0) + 4\sigma(1) + \cdots + 2^n\sigma(n-1)$.

Then a tree is just a particular kind of subset of $\mathbb{N}$.

Thus we can work in second-order arithmetic.

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But what about multiple instances?

Turing Ideals

A Turing ideal is an $\mathcal{I} \subseteq 2^{\mathbb{N}}$ s.t. if $B_1, \ldots, B_n \in \mathcal{I}$ and A is computable relative to $B_1, \ldots, B_n$ then $A \in \mathcal{I}$.

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A statement Φ of second-order arithmetic holds in $\mathcal{I}$ if Φ is true when $\exists X$ and $\forall X$ are replaced by $\exists X \in \mathcal{I}$ and $\forall X \in \mathcal{I}$.

Building Turing Ideals

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The proof uses the relativized Low Basis Theorem: If the binary tree T is computable relative to X then T has a path P s.t. $(P \oplus X)' \leqslant_{\mathbf{T}} X'$.

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Thm (Yu and Simpson). WWKL $\nvDash_{\mathbf{c}}$ WKL.

The proof uses the theory of algorithmic randomness.

Computable equivalence

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Computably equivalent to BKL (i.e., computably true):

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Computably equivalent to WKL:

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- the existence of maximal ideals for commutative rings
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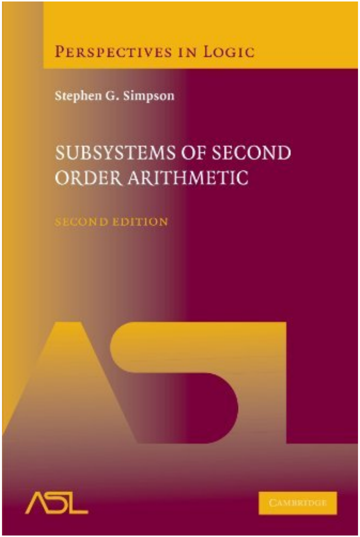
Part III: Reverse Mathematics

Stephen G. Simpson, *Subsystems of Second Order Arithmetic*



Second-Order Arithmetic and RCA_0

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Reverse Mathematics: fix a weak base system and calibrate the strength of principles by considering implications over this system.

Often in terms of a few subsystems of second-order arithmetic.

Full second-order arithmetic consists of

- axioms for a discrete ordered commutative semiring
- comprehension:

$$\exists X\, \forall n\, [n \in X \;\leftrightarrow\; \varphi(n)]$$

  for all formulas φ s.t. X is not free in φ

- induction:

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We obtain subsystems by limiting comprehension and induction.

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A bounded quantifier is one of the form $\forall x < t$ or $\exists x < t$.

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$$\exists x_1\ \forall x_2\ \exists x_3\ \forall x_4 \cdots Q x_n\ \varphi,$$

where φ is a bounded-quantifier formula and Q is $\exists$ if n is odd and $\forall$ if n is even.

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These formulas can have free variables.

The Weak Base System RCA_0

RCA_0 is obtained by restricting:

- comprehension to Δ^0_1-comprehension:

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This choice of base system creates a tight connection between this approach and computable entailment.

Some Equivalences over RCA_0

Provable in RCA_0

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Provably equivalent to KL over RCA_0:

- the existence of maximal ideals for commutative rings
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Computability and Definability

The Arithmetic Hierarchy

A first-order formula is one with no set variables.

$A \subseteq \mathbb{N}$ is defined in $\mathbb{N}$ by a first-order formula $\varphi(y)$ if: $k \in A$ iff $\varphi(k)$ holds in $\mathbb{N}$.

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A set is Σ^0_n if it is defined in $\mathbb{N}$ by some Σ^0_n first-order formula.

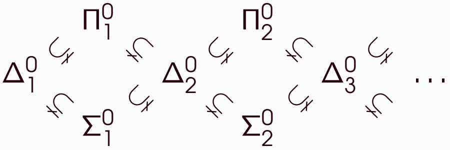
A set is Π^0_n if it is defined in $\mathbb{N}$ by some Π^0_n first-order formula.

A set is Δ^0_n if it is both Σ^0_n and Π^0_n.

A set is arithmetic if it is in one of these classes.

$$\begin{array}{ccccccccccc}
 & \subsetneq & \Pi^0_1 & \subsetneq & & \subsetneq & \Pi^0_2 & \subsetneq & & \subsetneq & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & \cdots \\
 & \subsetneq & \Sigma^0_1 & \subsetneq & & \subsetneq & \Sigma^0_2 & \subsetneq & & \subsetneq &
\end{array}$$

The Arithmetic Hierarchy



Thm (Kleene). A is Σ^0_1 iff A is c.e. Thus A is Δ^0_1 iff A is computable.

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Thm (Kleene). A is Σ^0_1 iff A is c.e. Thus A is Δ^0_1 iff A is computable.

Recall that Z' is the Halting Problem relative to Z.

Define $X^{(n)}$ as follows: $X^{(0)} = X$ and $X^{(n+1)} = (X^{(n)})'$.

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Thm (Post). A set is Σ^0_{n+1} iff it is c.e. relative to $\emptyset^{(n)}$, and is Δ^0_{n+1} iff it is computable relative to $\emptyset^{(n)}$.

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A set is Σ_n^0 relative to S if it is defined in $(\mathbb{N}, S)$ by some Σ_n^0 formula.

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Post's Theorem holds in relativized form.

In particular, A is Δ^0_1 relative to S iff A is computable relative to S.

Recall that RCA_0 is obtained by restricting:

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Δ^0_1-comprehension is (relative) computable comprehension.

Indeed, RCA stands for Recursive Comprehension Axiom.

A model in the language of second-order arithmetic consists of a first-order part $\mathcal{N} = (N; 0_N, 1_N, S_N, <_N, +_N, \cdot_N)$ and a second-order part $\mathcal{S} \subseteq 2^N$.

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Thm (Friedman). $\mathcal{S}$ is an ω-model of RCA_0 iff $\mathcal{S}$ is a Turing ideal.

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The converse does not always hold because non-ω-models of RCA_0 exist, but it often does.

The Reverse-Mathematical Universe

Mathematics in RCA_0

Several theorems can be proved in RCA_0, e.g. many basic properties of the natural numbers and the reals, as well as

- the existence of algebraic closures of fields
- Gödel's Completeness Theorem for theories
- the Intermediate Value Theorem
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Thm (Yokoyama). BKL is not provable in RCA_0.

Other Subsystems of Second-Order Arithmetic

ACA_0: RCA_0 + arithmetic comprehension:

$$\exists X \forall n [n \in X \leftrightarrow \varphi(n)]$$

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- the existence of maximal ideals for commutative rings
- the existence of bases for vector spaces
- the Bolzano-Weierstraß Theorem
- the existence of the Turing jump

  ⋮

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Equivalents of WKL_0

- the uniqueness of algebraic closures for fields
- the existence of prime ideals for commutative rings
- the Compactness Theorem for first-order logic
- the Extreme Value Theorem
- Brouwer's Fixed Point Theorem

⋮

WWKL_0: RCA_0 + Weak Weak König's Lemma

Equivalents of WWKL_0:

- the Vitali Covering Theorem
- the monotone convergence theorem for Lebesgue measure on [0, 1]
- the existence of (relatively) Martin-Löf random sequences

⋮

ATR_0: RCA_0 + arithmetic transfinite recursion

Other Subsystems of Second-Order Arithmetic

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Equivalents of ATR_0

- comparability of well-orderings
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Equivalents of Π^1_1-CA_0

- every countable Abelian group is the direct sum of a divisible group and a reduced group
- the Cantor-Bendixson Theorem

  ⋮

Relationships between Subsystems of Second-Order Arithmetic

$$\Pi^1_1\text{-}\mathrm{CA}_0$$

$$\downarrow$$

$$\mathrm{ATR}_0$$

$$\downarrow$$

$$\mathrm{ACA}_0$$

$$\downarrow$$

$$\mathrm{WKL}_0$$

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$$\mathrm{WWKL}_0$$

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$$\mathrm{RCA}_0$$

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$\mathrm{RT}^n_{<\infty}$ is $\forall k\ \mathrm{RT}^n_k$ and RT is $\forall n\ \forall k\ \mathrm{RT}^n_k$.

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Some consequences of RT^2_2

Ascending / Descending Sequence Principle (ADS): Every infinite linear order has an infinite ascending or descending sequence.

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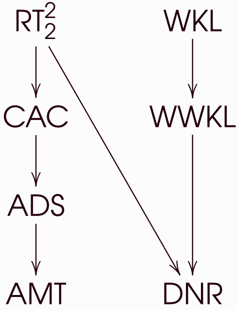
Atomic Model Theorem (AMT): Every complete atomic theory has an atomic model.

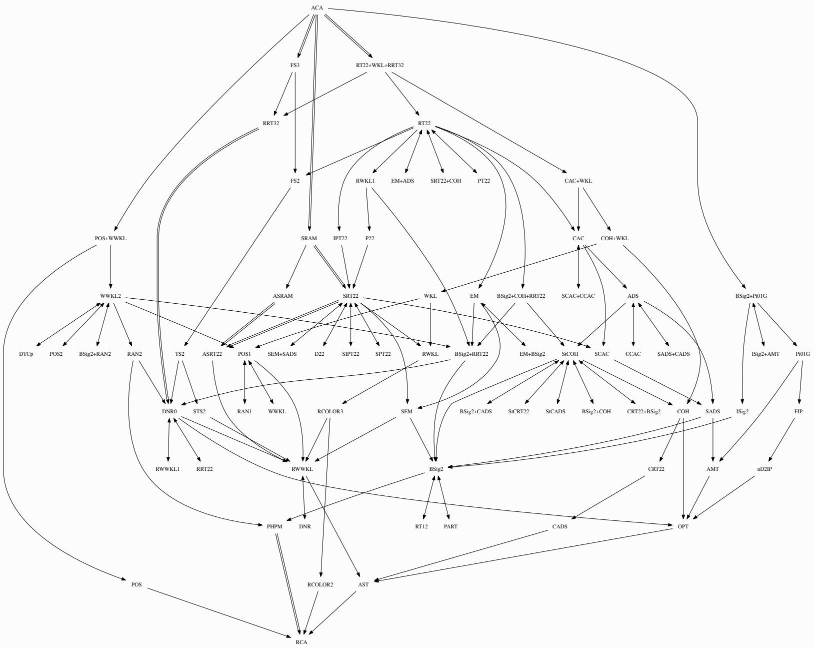
Existence of Diagonally Nonrecursive Functions (DNR): For every X, there is a function f s.t. $f(e) \neq \Phi_e^X(e)$ for all e.

A Small Part of the Universe Between RCA_0 and ACA_0



Combined results of Yu and Simpson; Giusto and Simpson; Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman; Hirschfeldt and Shore; Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman; Hirschfeldt, Shore, and Slaman; Liu; and Lerman, Solomon, and Towsner.

A Larger Part of the Universe Between RCA_0 and ACA_0



Tutorial Series on Reverse Mathematics

Denis R. Hirschfeldt — University of Chicago

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