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Orders on Computable Structures

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- Magma is a nonempty set with a binary operation: $(M, \cdot)$

- A *left-order* on the structure $(M, \cdot)$ is
  a linear ordering $<$ of the domain M, which
  it is left invariant with respect to $\cdot$
  $(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y]$

- $<$ is a *bi-order* (*order*) on the structure if
  $(\forall x, y, z)[x < y \Rightarrow (z \cdot x < z \cdot y \wedge x \cdot z < y \cdot z)]$

- $LO(M)$ the set of left orders on M
  $RO(M)$ the set of right orders on M
  $BiO(M)$ the set of bi-orders on M

- Given a left order $<_l$ on a group G, we have a right order $<_r$
  $x <_r y \Leftrightarrow y^{-1} <_l x^{-1}$

- G is left-orderable group $\Rightarrow G$ is *torsion-free*
  torsion-free: $(\forall x \in G - \{e\})[order(x) = \infty]$
  $e < x \Rightarrow x < x^2 < \cdots < x^n$

- (Levy)
  G is abelian and torsion-free $\Rightarrow G$ is orderable

  (Kokorin and Kopytov)
  Every torsion-free nilpotent group is orderable.

- Torsion-free, but not left-orderable group:
  $G = \langle x, y \mid xy^2x^{-1}y^2 = e,\ \ yx^2y^{-1}x^2 = e \rangle$

- $(\mathbb{Z}, +)$ has two orders, both computable.

  $(\mathbb{Z}^2, +)$ has $2^{\aleph_0}$ orders.

  $\mathbb{Z}^\omega = \bigoplus_{i\in\omega} \mathbb{Z}$, the direct sum of ω copies of $\mathbb{Z}$

  $(\mathbb{Z}^\omega, +)$ has $2^{\aleph_0}$ orders.

- (Solomon, 2002)

  (i) A computable torsion-free abelian group of finite rank $n > 1$ has an order in every Turing degree.

  (ii) A computable torsion-free abelian group of infinite rank has an order in every Turing degree $\geq \mathbf{0}'$.

  (iii) A computable torsion-free properly n-step nilpotent group G has an order in every Turing degree $\geq \mathbf{0^{(n)}}$.

- (Downey and Kurtz, 1986)

  There is a computable torsion-free abelian group G (hence orderable) such that G has no computable order.

- (Dobrica, 1983)

  Every computable torsion-free abelian group is isomorphic to a computable group with a computable basis (hence with a computable order).

- (Harrison-Trainor, to appear)

  There is a computable left-orderable group that is not isomorphic to a computable group with a computable left-order.

  Not known for the case of bi-orderable groups.

- (Solomon, 1998)
  For every bi-orderable computable group G, there is
  a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection
  from $BiO(G)$ to the set of all infinite paths of $\mathcal{T}$.

- Hence, by the *Low Basis Theorem* of Jockusch and Soare,
  $BiO(G)$ contains an order of *low* Turing degree.

  Also, $LO(G)$ contains an order of *low* Turing degree.

- For a bi-order $<$: $(a < b \wedge c < d \Rightarrow a \cdot c < b \cdot d)$
  $a < b \Rightarrow a \cdot c < b \cdot c$
  $c < d \Rightarrow b \cdot c < b \cdot d$

- Not necessarily true for a left order.

  Example: Klein bottle group $K = \left\langle a, b \mid a^{-1}ba = b^{-1} \right\rangle$

  Left-orderable, but not bi-orderable
  $b > e \Leftrightarrow a^{-1}ba = b^{-1} > e$

  $ba = ab^{-1}$

  b and a^2 commute:
  $a^2b = a^2(a^{-1}b^{-1}a) = ab^{-1}a = ba^2$

  $ba \neq a$ but $(ba)^2 = baba = ab^{-1}ba = a^2$

- A magma $(Q, *)$ is called a *quandle* if:

  1. $(\forall a)[a * a = a]$ (idempotence);

  2. for every $b \in Q$, the mapping $*_b : Q \to Q$ defined by

  $$*_b(a) = a * b$$

  is bijective;

  3. $(\forall a, b, c)[(a * b) * c = (a * c) * (b * c)]$ (right self-distributivity).

- A quandle Q is called *trivial* if the operation $*$ is defined by

  $$(\forall a, b)[a * b = a].$$

  Every linear ordering of elements of Q is right invariant.

- For a group G, the *conjugate* quandle Conj(G) is one with domain G and the operation $*$ given by $a * b = b^{-1}ab$.

  Then every bi-order on G induces a right order on Conj(G).

  Let P be a bi-order on G.

  Using P, we define R on Conj(G) as

$$(\forall a, b)[(a, b) \in R \Leftrightarrow (a, b) \in P]$$

  The order R is right invariant because for $(a, b) \in R$ and $c \in$ Conj(G):

$$(a, b) \in P \Rightarrow (c^{-1}ac, c^{-1}bc) \in P \Rightarrow (a * c, b * c) \in R$$

- Not all right orders on Conj(G) are induced by bi-orders on G.
  It is possible to have $BiO(G) = \emptyset$,
  while $RO(\text{Conj}(G)) \neq \emptyset$.

  Let G be an abelian group with torsion.
  Then $BiO(G) = \emptyset$, but Conj(G) is a trivial quandle,
  so it admits many right orders.

- *Topology* defined on $LO(M)$ by subbasis $\{S_{(a,b)}\}_{(a,b)\in(M\times M)-\mathbf{\Delta}}$ where $\mathbf{\Delta} = \{(a,a) \mid a \in M\}$:

$$S_{(a,b)} = \{R \in LO(M) \mid (a,b) \in R\}.$$

- (Dabkowska, Dabkowski, Harizanov, Przytycki, and Veve, 2007)

  Let M be a left-orderable magma with cardinality $|M| = \mathfrak{m} \geq \aleph_0$. Then $LO(M)$ is a compact space. $BiO(M)$ is also a compact space.

  By Vedenissoff's theorem, $LO(M)$ can be homeomorphically embedded into the Cantor cube $\{0,1\}^{\mathfrak{m}}$. Moreover, $LO(M)$ is a closed subspace of the Cantor cube $\{0,1\}^{\mathfrak{m}}$.

- If M is a countable magma, then $LO(M)$ is metrizable.

- If $M = G$ is a group, we showed how we could also use Conrad's theorem to establish that $LO(G)$ is compact.

- (Conrad, 1959)
  A partial left order given by its positive cone P can be extended to a total left order on G iff for every $\{x_1, ..., x_n\} \subseteq G \backslash \{e\}$ there are $\epsilon_1, ..., \epsilon_n$, $\epsilon_i \in \{1, -1\}$, such that

$$e \notin sgr((P \backslash \{e\}) \cup \{x_1^{\epsilon_1}, ..., x_n^{\epsilon_n}\}),$$

  where $sgr(A)$ is the sub-semigroup of G generated by A.

- For a countable group G, $LO(G) \neq \emptyset$ is homeomorphic to the Cantor set iff for any sequence $(a_0, b_0), ..., (a_{k-1}, b_{k-1})$, $S_{(a_0,b_0)} \cap \cdots \cap S_{(a_{k-1},b_{k-1})}$ is either empty or infinite.

- (Sikora, 2004)
  The space $LO(\mathbb{Z}^n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Dabkowska, 2006)
  The space $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.

- There are countable groups with infinitely countably many bi-orders.

- (Linnell, 2006)
  The space of left orders of a countable left-orderable group is either finite or contains a homeomorphic copy of the Cantor set.

- Let $F_n = \langle x_0, x_1, \ldots, x_{n-1} \mid \ \rangle$ be a free group of rank n.

- *Conjecture* (Sikora, 2004)
  For $n > 1$, the spaces $LO(F_n)$ and $BiO(F_n)$ are homeomorphic to the Cantor set.

- (Navas-Flores, 2008)
  The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

  Not known for $BiO(F_n)$ for $n > 1$.

- (Dabkowska, Dabkowski, Harizanov, Togha, 2010)
  For a computable group G isomorphic to a free group F_n of rank $n > 1$, we have a bi-order in every Turing degree.

Proof sketch:

For a group G, the *lower central series* is the descending sequence of subgroups $(\gamma_\alpha(G))_\alpha$ defined as:

$$\begin{aligned}
&\gamma_1(G) = G, \\
&\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G], \\
&\gamma_\beta(G) = \textstyle\bigcap_{\alpha<\beta}\ \gamma_\alpha(G), \text{ when } \beta \text{ is a limit ordinal,}
\end{aligned}$$

where $[A, B]$ is the subgroup of G generated by the elements $a^{-1}b^{-1}ab$, with $a \in A$ and $b \in B$.

- *Lower central series* of F_n: $\gamma_1(F_n) \geq \cdots \geq \gamma_i(F_n) \geq \cdots$

- (Magnus) $\gamma_\omega = \displaystyle\bigcap_{i=1}^{\omega} \gamma_i(F_n) = \{e\}$

- (Hall) $\gamma_i(F_n)/\gamma_{i+1}(F_n) \cong \mathbb{Z}^{k_i}$,
  where $k_i = \frac{1}{i}\sum_{d|i} \mu(\frac{i}{d})n^d$, μ Möbius function

- Isomorphism uniformly computable since a basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$ can be found algorithmically in n, i.

- Construct bi-orders on F_n using bi-orders on $\gamma_i(F_n)/\gamma_{i+1}(F_n)$.

- Different choices of orders on quotients induce different orders on F_n.

- Produce a bi-order on F_n of a given Turing degree.

- (Harizanov, Knight, McCoy, Puzarenko, Solomon and Wallbaum, preprint)

  Let $F_\infty = \langle x_0, x_1, \ldots \mid \ \rangle$ be a free group of rank $\aleph_0$.
  There is a computable copy F of F_∞
  with no computable left order (hence no computable bi-order).

- *Corollary*
  The space $LO(F_\infty)$ and the space $BiO(F_\infty)$ are homeomorphic to the Cantor set.

- For every left-orderable computable group G, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $LO(G)$ to the set of all infinite paths of $\mathcal{T}$.

- An isolated path in a computable binary tree must be computable.

  In a computable binary tree with infinite paths and no computable ones, the space of paths is homeomorphic to the Cantor set.

  Topology of the space of orders does not change under isomorphisms, so it is the same for the whole isomorphism class.

- We can generalize the construction for free groups of finite rank > 1 to a class of finitely presented, *residually nilpotent* groups that are not nilpotent.

- (Chubb, Dabkowski, and Harizanov, to appear in 2017)
Let G be a finitely presented, torsion-free, computable group.
Let $G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$ be the lower central series of G.
If $\gamma_\omega(G) = \{e\}$, and
$\gamma_i(G)/\gamma_{i+1}(G)$ is nontrivial and torsion-free for each $i = 1, 2, \ldots$,
then there is a bi-order on G in every tt-degree.

- *Example*: Surface groups of genus $n > 1$:
$\langle x_1, y_1, \ldots, x_n, y_n \mid [x_1, y_1] \cdots [x_n, y_n] = e \rangle$

HAPPY BIRTHDAY, ROD!