1-Generic Degrees Bounding Minimal Degrees—*Revisited*

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  - Week 2: Theodore A Slaman
  - Week 3: W Hugh Woodin
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n-Generic degrees and minimal degrees

- No minimal degree bounds an n-generic degree.
- (Jockusch 1980) For $n \geq 2$, no n-generic degree bounds a minimal degree.
- (Chong and Jockusch 1984) No 1-generic degree $< \mathbf{0}'$ bounds a minimal degree.
- (Haught 1986) Every degree below a 1-generic degree $< \mathbf{0}'$ is 1-generic.

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Constructing a minimal degree

Typical construction of a set of minimal degree applies the Spector “tree construction" method:

Given Φ_e and an infinite recursive perfect tree $T \subset 2^{<\omega}$, define by recursion a *splitting subree* $Sp(e, T) \subset T$:

- If $\tau_1, \tau_2 \in Sp(e, T)$ are incomparable, then $\Phi_e^{\tau_1}(x) \neq \Phi_e^{\tau_2}(x)$ for some x.

There are two possibilities:

1. (Splitting tree) Every $\tau \in Sp(e, T)$ has a (least) pair of incomparable extensions in $Sp(e, T)$. Let $T_e = Sp(e, T)$. Then if $X \in [T_e]$, $\Phi_e^X \equiv_T X$;
2. (Full tree) There is a $\tau \in Sp(e, T)$ with no extension in $Sp(e, T)$. Let $T_e = \{\tau' \in T : \tau' \succeq \tau\}$. Then any $X \in [T_e]$ satisfies Φ_e^X is partial or Φ_e^X is recursive.

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The Tree Method

- Starting with $T = 2^{<\omega}$, define $T_0 \supset T_1 \supset \cdots$ so that T_{e+1} is a splitting or full subtree of T_e.
- Any $X \in \bigcap_e [T_e]$ has minimal degree. There is an $X <_T \emptyset''$.
- The split into (1) or (2) is a $\emptyset''$-decision.
- Σ^0_2 induction is sufficient to implement the Spector construction.

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Models of $P^- + I\Sigma^0_1 + \neg I\Sigma^0_2$

Fix $\mathfrak{M} = (M, +, \cdot, 0, 1) \models P^- + I\Sigma^0_1 + \neg I\Sigma^0_2$.

- (Tame cut) There is a Σ^0_2 cut I with a Σ^0_2 increasing, cofinal $g : I \to M$.
- Let $I < a$. For $i \leq a$, define

$$\Phi^{\sigma}_i(x) = \begin{cases} \sigma(x) & \text{if } x \leq |\sigma| \text{ and } g'(x, i) \neq g'(x + 1, i) \\ 0 & \text{otherwise} \end{cases}$$

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Tree Method in $\neg I\Sigma^0_2$

- Let $T = 2^{<M}$. Define splitting tree and full tree as before.
- Then for $i \in I$, T_i is a full tree with root of length $\geq g(i)$.
- T_i is *not* defined for $i \notin I$.

Hence the Spector method fails.

Question. Is there a set of minimal degree $<_T \emptyset'$ or $<_T \emptyset''$ in $\mathfrak{M}$?

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$X \subset M$ is *regular* if $X \upharpoonright s$ is $\mathfrak{M}$-finite for every $s \in M$.

- (Chong and Mourad 1990) There is an $\mathfrak{M} \models B\Sigma^0_2$ in which $\omega = I$ is a Σ^0_2-cut of minimal degree.
- $I <_T \emptyset''$ and nonregular.
- If $\mathfrak{M} \models I\Sigma^0_1$ is countable, then there is a regular set X of minimal degree. But X need not be definable.

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Minimal Degrees in $\neg I\Sigma_2$

Theorem

Let $\mathfrak{M} \models P^- + B\Sigma_2^0 + \neg I\Sigma_2^0$. if $X <_T \emptyset''$ is regular and has minimal degree, then $X <_T \emptyset'$ and $X' \equiv_T \emptyset'$.

Theorem

RCA_0 + *"There is a minimal degree" does not imply $B\Sigma_2^0$.*

Question: Is there a model of

$$\mathrm{RCA}_0 + B\Sigma_2^0 + \neg I\Sigma_2^0 + \text{“There is a minimal degree”?}$$

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Minimal Degree in $\neg I\Sigma_2^0$

Theorem

The following are equivalent over the base theory $P^- + B\Sigma_2^0$:

1. $\mathfrak{M} \models P^- + I\Sigma_2^0$
2. $$\mathfrak{M} \models \text{“There is a 1-generic degree} < \mathbf{0}'' \text{ bounding a minimal degree”}$$

Sketch of proof

(1) $\Rightarrow$ (2) follows the proof for ω.

(2) $\Rightarrow$ (1):

Let $G <_T \emptyset''$ be 1-generic in $\mathfrak{M} \models P^- + B\Sigma_2^0 + \neg I\Sigma_2^0$;

Fact. (Chong and Yang 2006) If $\mathfrak{M} \models P^- + B\Sigma_2^0 + \neg I\Sigma_2^0$ with a Σ_2^0-cut I, then every regular $X \leq_T \emptyset''$ is recursive in $I \oplus \emptyset'$.

Suppose $\emptyset <_T B \leq_T G \leq_T I \oplus \emptyset'$. Construct a 1-generic $D \leq_T B$ to conclude that B is not a set of minimal degree.

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Second order arithmetic

There is much more restriction on set existence over RCA_0:

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Let $\mathfrak{M} = (M, S)$. If $\mathfrak{M} \models \mathrm{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$, then every Σ^0_3-definable $X \in S$ (without set parameters) is low.

Corollary

If $\mathfrak{M}$ is as above, then no Σ^0_3-definable 1-generic set in $\mathfrak{M}$ bounds a set of minimal degree.

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