The Complexity of Complexity

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RodFest, January 5, 2017

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The Complexity of Complexity
New Insights on the (Non)hardness of Circuit Minimization and Related Problems

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Joint work with
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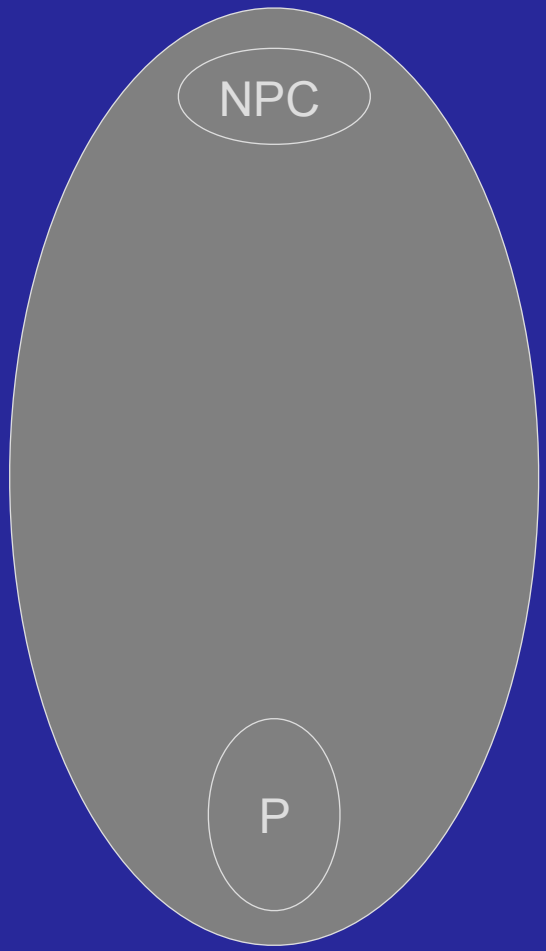
NP-Intermediate Problems



NP-Intermediate Problems

Ladner (1975) showed that intermediate problems exist unless P=NP.

But these problems are "unnatural".

NP-Intermediate Problems

Some prominent candidates were studied early on…

NP-Intermediate Problems

Some prominent candidates were studied early on…

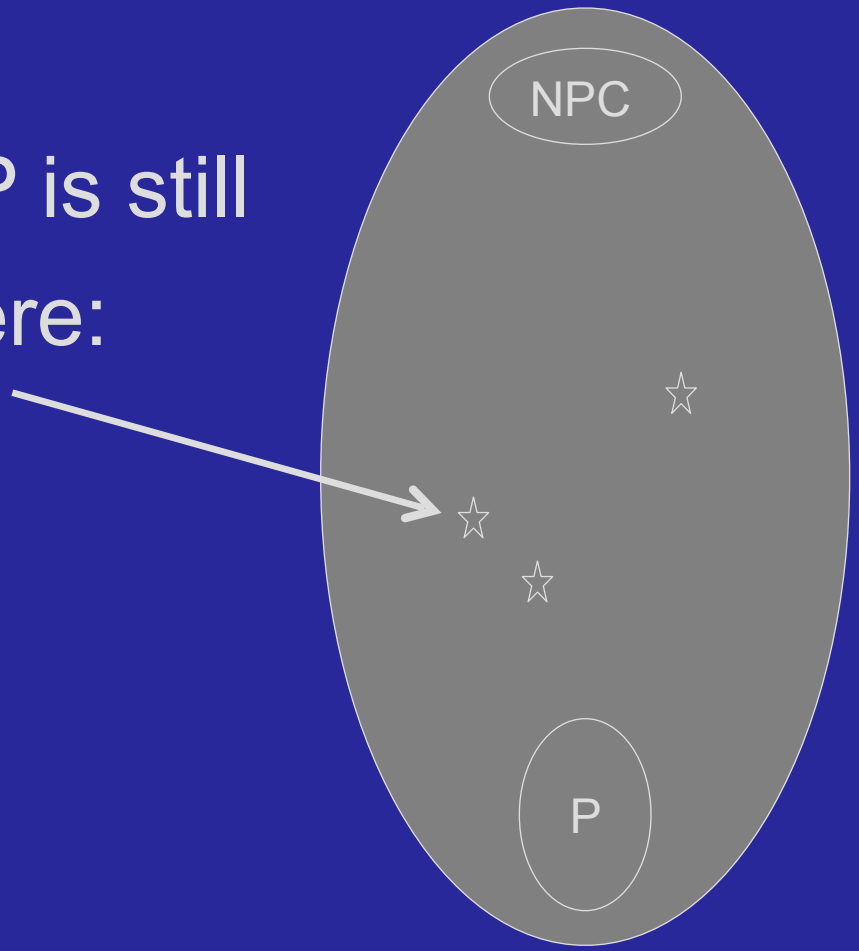
Graph Genus

Linear Programming

…but subsequently "moved".

NP-Intermediate Problems

MCSP is still here:



The Context

- The Minimum Circuit Size Problem (MCSP) = {(f,i) : f is the truth-table of a function that has a circuit of size ≤ i}.
- That is, MCSP is the overgraph of the "circuit size" complexity measure SIZE(f).

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The Context

- The Minimum Circuit Size Problem (MCSP) = {(f,i) : f is the truth-table of a function that has a circuit of size ≤ i}.
- In NP, but not known (or widely believed) to be NP-complete. [Kabanets, Cai], [Murray, Williams], [A, Holden, Kabanets], [Hirahara, Watanabe]
- Lots of reasons to believe it's not in P.

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More Context

- MCSP is more like a family of problems, than a single problem.
- For instance "size" could mean "# of wires" or "# of gates", or "# of bits to describe the circuit", etc.
- None of these is known to be reducible to any other – but all can stand in for "MCSP".
- One more such variant:
  $O_{KT} = \{(x,i) : KT(x) \leq i\}$

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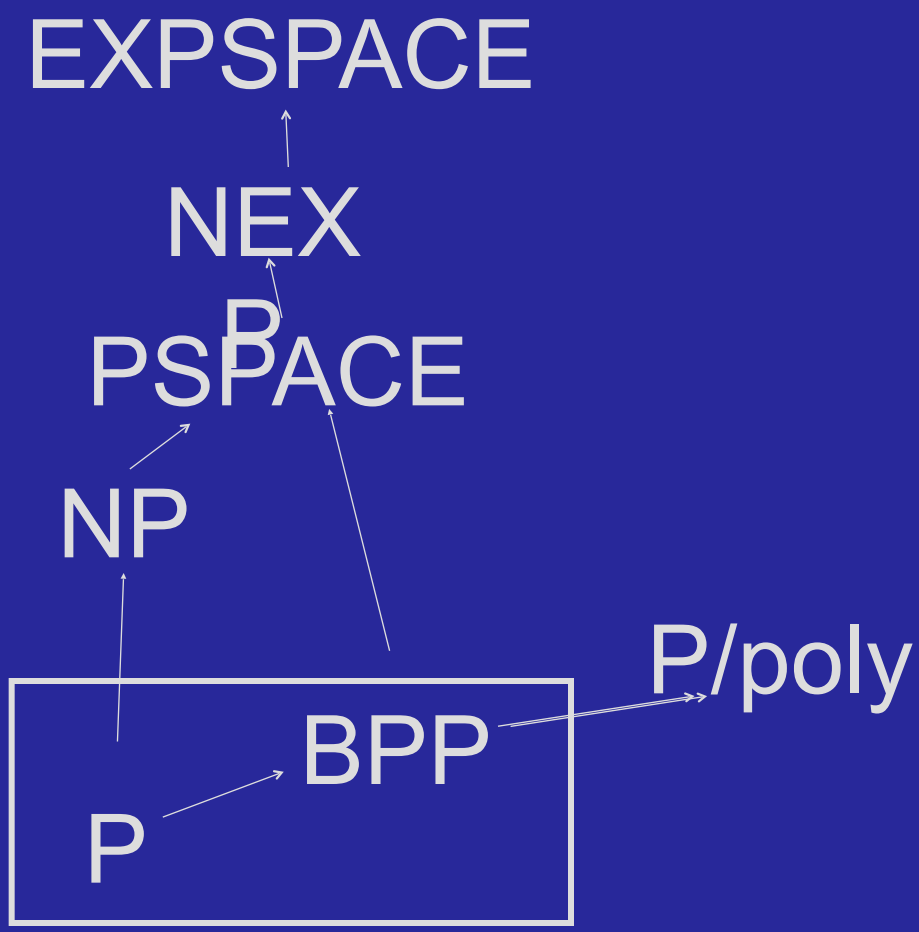
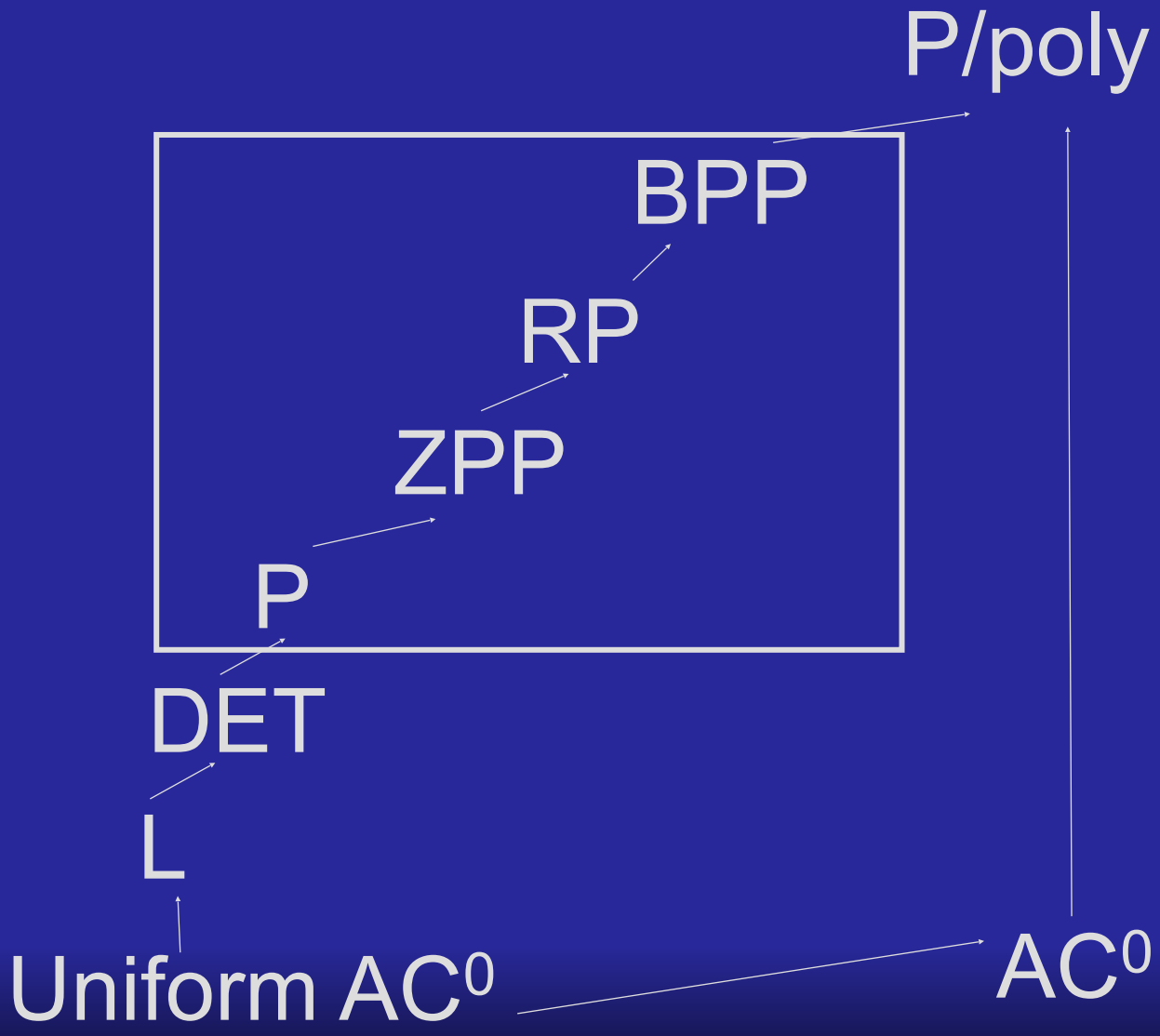
Complexity Classes



More Complexity Classes



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Yet More Context

- Factoring and Discrete Log are in ZPP^{MCSP}.
- Graph Isomorphism is in RP^{MCSP}.
- Every promise problem in SZK is in (Promise) BPP^{MCSP}.

- But is MCSP NP-hard?

Statistical Zero Knowledge
SZK is contained in
NP/poly ∩ coNP/poly

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Yet More Context

- Factoring and Discrete Log are in ZPP^{MCSP}.
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- But is MCSP NP-hard?

All these hardness results use probabilistic reductions.

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Consequences of Hardness

- From [MW],[AHK], we know:
- MCSP NP-hard under $\leq_m^p$ implies EXP ≠ ZPP.
- MCSP NP-hard under $\leq_m^{\log}$ implies PSPACE ≠ ZPP.
- MCSP NP-hard under $\leq_m^{AC^0}$ implies BPP= P, NP not in P/poly, DSPACE(n) not in SIZE($2^{\varepsilon n}$)
- MCSP NP-hard under TIME($\sqrt[3]{n}$) local reductions implies FALSE

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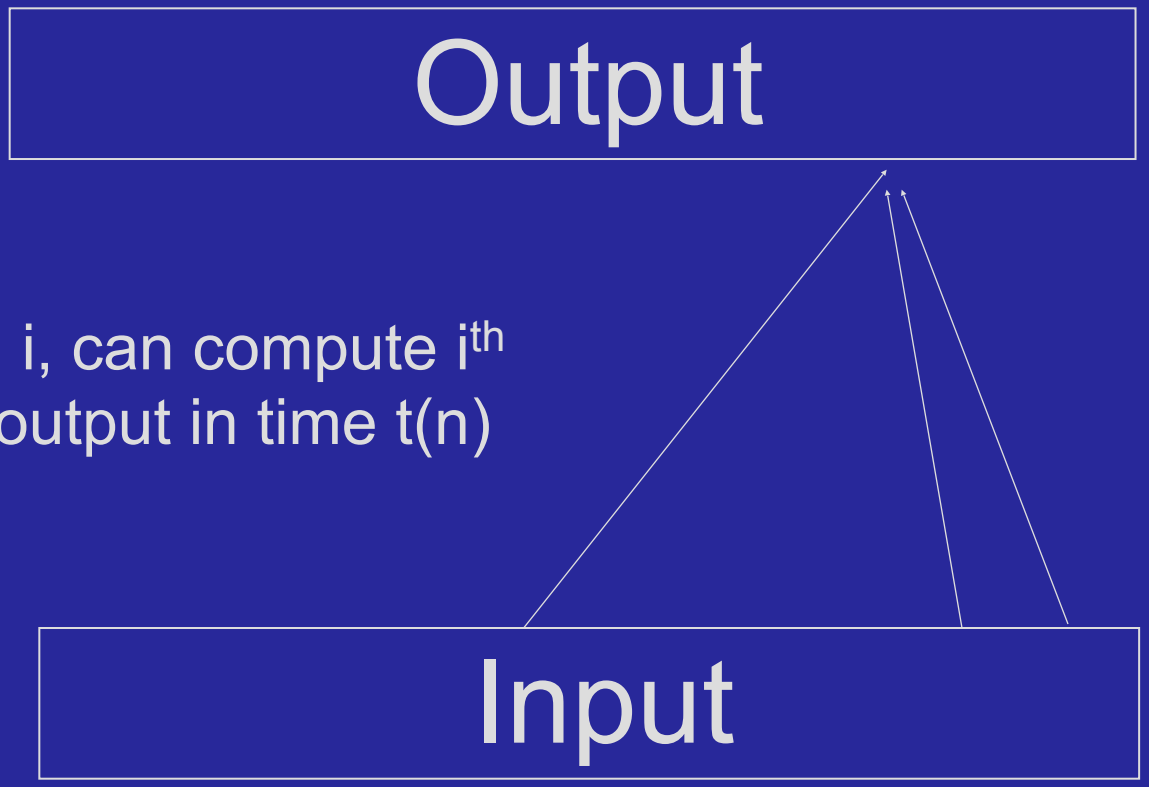
Local Reductions



Given i, can compute i^{th} bit of output in time t(n)

Local Reductions

- SAT is NP-complete under TIME(log n) local reductions.
- PARITY does *not* reduce to MCSP under TIME($\sqrt[3]{n}$) local reductions.
- How about NC^0 reductions?
  - For MCSP we still don't know.
  - For O_{KT}, we have a theorem:
  - O_{KT} *is hard for* DET under non-uniform NC^0 reductions.

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First Theorem

- O_{KT} is hard for DET under non-uniform NC^0 reductions.
- Corollary: O_{KT} is not in $AC^0[p]$ for any prime p.
- Circuit lower bounds imply O_{KT} is hard for DET under *uniform* AC^0–Turing reductions.
- O_{KT} hard for PARITY under these same uniform reductions implies (similar) circuit lower bounds.

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First Theorem

- O_{KT} is hard for DET under non-uniform NC^0 reductions.
- Corollary: O_{KT} is not in $AC^0[p]$ for any prime p.
- Independently, [Oliveira & Santhanam] have shown that DET reduces to MCSP via non-uniform TC^0 reductions. (This still leaves open the question of whether MCSP is in $AC^0[p]$.)

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First Theorem

- O_{KT} is hard for DET under non-uniform NC^0 reductions.
- Corollary: O_K and O_C are hard for DET under non-uniform NC^0 reductions.
- Open question: Is O_K or O_C hard for P or hard for Σ^0_1 under non-uniform AC^0 or NC^0 $\leq_m$ reductions?
- Non-uniformity is key. No uniform $\leq_m$ reduction can reduce anything nontrivial to O_K or O_C.

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First Theorem

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- Corollary: O_K and O_C are hard for DET under non-uniform NC^0 reductions.
- Open question: Is O_K or O_C hard for P or hard for Σ^0_1 under non-uniform AC^0 or NC^0 $\leq_m$ reductions?
- O_K and O_C are hard for Σ^0_1 under (non-uniform) P/poly-Turing reductions [ABKMR].

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Three Bizarre Inclusions

- NEXP is contained in NP^{O_K} (for every U).

- PSPACE is contained in P^{O_K} (for every U).
- BPP is contained in $P_{tt}^{O_K}$ (for every U).
  - The decidable sets that are in $P_{tt}^{O_K}$ for every U are in PSPACE. [AFG11]
  - [CDELM] The sets that are in $P_{tt}^{O_K}$ for every U are decidable.

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Three Bizarre Inclusions

- NEXP is contained in NP^{O_K} (for every U).
  - The sets that are in NP^{O_K} for every U are in EXPSPACE.
- PSPACE is contained in P^{O_K} (for every U).
- BPP is contained in $P_{tt}^{O_K}$ (for every U).
  - The sets that are in $P_{tt}^{O_K}$ for every U are in PSPACE.
- Thus there are reasons to care about efficient reductions to O_K and O_C.

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Context for the 2nd Theorem

- Recall: Every promise problem in SZK is in (Promise) BPP^{MCSP}.
- SZK is usually defined in terms of "promise problems".

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Promise Problems

Ordinary decision problems.

Promise Problems

A "solution"

Ordinary decision problems.

Promise Problems.

Promise Problems

- A promise problem (Y,N) is NP-hard if SAT reduces to *every* solution.
- $Gap_\varepsilon MCSP$:
  - $Y = \{(f,s) : SIZE(f) < s/|f|^{1-\varepsilon(|f|)}\}$
  - $N = \{(f,s) : SIZE(f) > s\}$
- Note: $Gap_\varepsilon MCSP$ becomes easier when ε is smaller.
- If $\varepsilon(n) = o(1)$, then $Gap_\varepsilon MCSP \in DTIME(2^{n^{o(1)}})$.
- Related to Natural Proofs for $SIZE(2^{o(n)})$.

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The 2nd Theorem

- If cryptographically-secure one-way functions exist, then there is a function $\varepsilon(n)=o(1)$ such that
  - $\text{Gap}_{\varepsilon}\text{MCSP}$ is not in P/poly, and
  - $\text{Gap}_{\varepsilon}\text{MCSP}$ is not NP-hard under P/poly-Turing reductions.

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The 2nd Theorem

- If cryptographically-secure one-way functions exist, then there is a function $\varepsilon(n)=o(1)$ such that
  - $\mathrm{Gap}_{\varepsilon}\mathrm{O}_{\mathrm{KT}}$ is not in P/poly, and
  - $\mathrm{Gap}_{\varepsilon}\mathrm{O}_{\mathrm{KT}}$ is not NP-hard under P/poly-Turing reductions.

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Three Bizarre Inclusions, Reprise

- NEXP is contained in NP^{GapO_K}, which is contained in EXPSPACE.
- PSPACE is contained in P^{GapO_K}.
- BPP is contained in $P_{tt}^{GapO_K}$, which is contained in PSPACE.
- I'm on record as conjecturing BPP = $P_{tt}^{O_K}$
- Might it be easier to show BPP = $P_{tt}^{GapO_K}$?
- In particular, is there an easier way than [CDELM] to show that $P_{tt}^{GapO_K}$ is decidable?

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1st theorem (O_{KT} hard for DET)

- [Toran] showed that DET AC^0-reduces to the isomorphism problem for rigid graphs.
- [A,Grochow,Moore] showed that rigid graph isomorphism is in $RP^{O_{KT}}$. We'll modify that proof.

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1st Theorem: Proof

- On input (G_0, G_1)
  - Randomly pick a bit string $w = w_1 w_2 \ldots w_t$.
  - Pick random permutations $\pi_1 \ldots \pi_t$.
  - Let $z = w\pi_1(G_{w_1})\pi_2(G_{w_2})\ldots\pi_t(G_{w_t})$
- If G_0 and G_1 are not isomorphic, then z allows us to reconstruct w and $\pi_1 \ldots \pi_t$, so that z has (non-time-bounded) K-complexity around t+ts (where s = log n!), whp. Hence KT(z) > t+ts.
- Otherwise, KT(z) is around n^2+ts.

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1st Theorem: Proof

- On input (G_0,G_1)
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  - Pick random permutations $\pi_1\ldots\pi_t$.
  - Let $z= w\pi_1(G_{w_1})\pi_2(G_{w_2})\ldots\pi_t(G_{w_t})$
- The mapping $(G_0,G_1,\pi_1,\pi_2,\ldots\pi_t) \rightarrow z$ is computable in AC^0 if the permutations are presented as a list of the form $(i,\pi(i))$.
- But a random bit string is *not* going to encode a sequence of permutations.

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1st Theorem: Proof

- On input (G_0, G_1)
  - Randomly pick a bit string $w = w_1 w_2 \ldots w_t$.
  - Pick random permutations $\pi_1 \ldots \pi_t$.
  - Let $z = w\pi_1(G_{w_1})\pi_2(G_{w_2})\ldots\pi_t(G_{w_t})$
- Solution: There is a logspace-computable function f that takes $(nt)^{O(1)}$ random bits and with high probability outputs a (nearly)-uniformly random sequence $\pi_1, \pi_2, \ldots \pi_t$.

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1st Theorem: Proof

- On input (G_0,G_1)
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  - Pick random permutations $\pi_1\ldots\pi_t$.
  - Let $z= w\pi_1(G_{w_1})\pi_2(G_{w_2})\ldots\pi_t(G_{w_t})$
- Thus there is an AC^0 function g such that $g(G_0,G_1,f(r))$ is a probabilistic reduction from Rigid Graph Isomorphism to O_{KT}. Hardwiring in a good choice of f(r) yields a non-uniform AC^0 reduction.

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1st Theorem: Proof

- On input (G_0, G_1)
  - Randomly pick a bit string $w = w_1 w_2 \ldots w_t$.
  - Pick random permutations $\pi_1 \ldots \pi_t$.
  - Let $z = w\pi_1(G_{w_1})\pi_2(G_{w_2})\ldots\pi_t(G_{w_t})$
- Combining this with [Toran], O_{KT} is hard for DET under AC^0 reductions.
- Now, by [Agrawal,A,Rudich], O_{KT} is also hard for DET under NC^0 reductions.

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Congratulations, Rod!

- *Thanks!*

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