Gravitational birefringence of light?

Christian Duval, Loïc Marsot & Thomas Schücker

to the memory of Christian Duval

3 axioms for general relativity

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} - \Lambda \, g_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

$$\not{m} \, \frac{D}{d\tau} \frac{d}{d\tau} x^\mu \qquad\qquad = 0$$

$$\Delta\tau = \oint \sqrt{g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp}} \, dp \, , \quad (+ - - -)$$

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$$\Downarrow \qquad \text{if test particle has no spin}$$

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Adding spin to the geodesic equation

$$\dot{x}^\mu = P^\mu \qquad\qquad \text{Einstein 1907 – 1915}$$

$$\dot{P}^\mu = 0$$

$P_\mu P^\mu =: m^2$ is conserved.

Adding spin to the geodesic equation

Now add spin $S^{[\mu\nu]}(\tau)$.

$$\dot{x}^\mu \neq P^\mu$$ Einstein 1907 – 1915

$$\dot{P}^\mu = 0 - \frac{1}{2} R^\mu{}_{\rho\alpha\beta} S^{\alpha\beta} \dot{x}^\rho$$ Mathisson 1937, Papapetrou 1951

$$\dot{S}^{\mu\nu} = P^\mu \dot{x}^\nu - P^\nu \dot{x}^\mu$$ Dixon 1970, Souriau 1974

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$\not\Rightarrow \quad P_\mu P^\mu =: m^2$ is conserved.

If $S^{\mu\nu} \mathrel{\mathring{=}} 0$ and if $\dot{x}^\mu$ and P^μ do not vanish, the spin evolution implies:

$$\dot{x}^\mu(\tau) = \gamma(\tau)\, P^\mu(\tau)$$

and we can reparameterize τ to achieve $\gamma(\tilde{\tau}) \mathrel{\mathring{=}} 1$.

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\dot{x}^\mu &\not= P^\mu && \text{Einstein 1907 – 1915} \\
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However, if $S^{\mu\nu} \neq 0$ we only have $4 + 6$ equations for $4 + 4 + 6$ unknowns and we must add 4 “supplementary conditions”.

2 main chapels

Pirani 1957: $S^{\mu}{}_{\nu}\dot{x}^{\nu} = 0$

- Massless particles move on geodesics, no birefringence of light.

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Tulczyjew 1959: $S^{\mu}{}_{\nu}P^{\nu} = 0$

- $P_{\mu}P^{\mu}$ is conserved unless $P_{\mu}P^{\mu} - \frac{1}{4}R_{\mu\nu\alpha\beta}S^{\mu\nu}S^{\alpha\beta} = 0$,
- $P_{\mu}\dot{x}^{\mu} = 0$,
- $\frac{1}{2}S_{\mu\nu}S^{\mu\nu} =: s^2$ is conserved.

After a $1+3$ split, the "spin scalar" s will become $s = \vec{s}\cdot\vec{p}/p =: \chi\hbar$ and $\vec{s} = s\,\vec{p}/p + \vec{s}^{\perp}$. Quantum Mechanics says: photons have "helicity" $\chi = \pm 1$ and $|\vec{s}^{\perp}| = \hbar$.

Equations of motion for massless test particles with spin

$$\dot{x}^\mu = P^\mu + 2\,\frac{S^\mu{}_\nu R^\nu{}_{\beta\rho\sigma} S^{\rho\sigma}}{R_{\alpha\beta\rho\sigma} S^{\alpha\beta} S^{\rho\sigma}}\, P^\beta$$

$$\dot{P}^\mu = -s\,\frac{\sqrt{-\det(R^\alpha{}_{\beta\rho\sigma} S^{\rho\sigma})}}{R_{\alpha\beta\rho\sigma} S^{\alpha\beta} S^{\rho\sigma}}\, P^\mu$$

$$\dot{S}^{\mu\nu} = P^\mu \dot{x}^\nu - P^\nu \dot{x}^\mu$$

Souriau 1974, Saturnini 1976

3 delicate features:

- No flat space limit
  (Non-vanishing cosmological constant Λ helps. Marsot 2020)
- No zero-spin limit
- Superluminal velocities: due to the anomalous velocity $\dot{x}^\mu$ is spacelike.

Birefringence of light in electric field with gradient

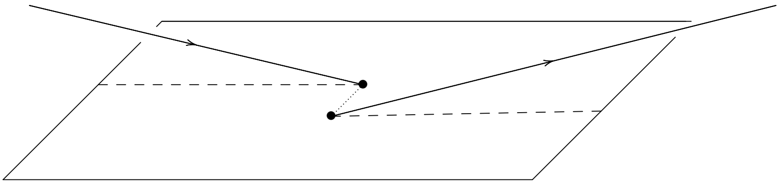

Figure: The Fedorov (1955) Imbert (1972) effect for reflection: A plane glass surface reflects an incoming, circularly polarized light beam. The dashed lines indicate the orthogonal projections of incoming and reflected light beams onto the glass surface. The dotted line (between the blobs) is the offset between incoming and reflected beams. It is of the order of the wavelength of the light beam.

Hosten & Kwiat, "Observation of the Spin Hall Effect of Light via weak measurements", Science **319** (2008) 787.

K. Bliokh, Niv, Kleinert & Hasman, "Geometrodynamics of spinning light", Nature Photonics **2** (2008) 748.

Birefringence in a Robertson-Walker metric

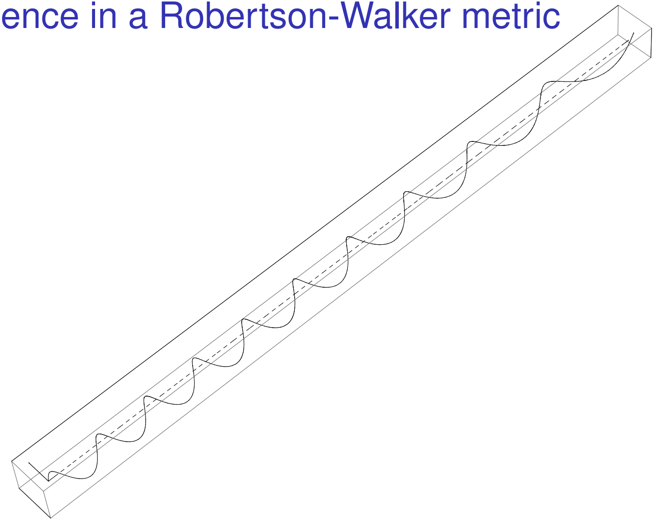

Figure: The trajectory of photons is the helix. The dashed line is the null geodesic. The transverse spin $\vec{s}_e^{\perp}$ at emission time t_e is indicated by the short arrow at the left.

- $R_{\text{helix}}(t) \sim \dfrac{a(t)}{a_e}\,\dfrac{\lambda_e}{2\pi} + O\left(\dfrac{\lambda_e}{2\pi\, a_e}\right)^2,$

- $T_{\text{helix}}(t) \sim \dfrac{a(t)}{a_e}\,\dfrac{\lambda_e}{1+Q},\ Q(t) := \dfrac{-aa''}{a'^2+K},\ K := \frac{1}{6}\,{}^{(3)}R$

- The projection of the photon on the geodesic moves with the speed of light c $(=1)$. Therefore the speed of the photon on the helix is $\sim \sqrt{2}\,c$.

- The spin vector $\vec{s}$ rotates with the same period $T_{\text{helix}}(t)$.

!! $|\vec{s}^{\perp}|$ is not conserved, but $|\vec{s}^{\perp}|\sqrt{a'^2+K}$ is.

Duval & S 2017

Birefringence in the Schwarzschild metric

Saturnini 1976; Duval, Marsot & S 2017

Let us take initial conditions $t = 0$ at the perihelion:

$$\vec{x}_0 = \begin{pmatrix} r_0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{p}_0 = \begin{pmatrix} 0 \\ p_0 \\ 0 \end{pmatrix} \quad \vec{s}_0 = \begin{pmatrix} 0 \\ s \\ s_0^\perp \end{pmatrix}$$

!! Stable numerical and perturbative solutions exist only if $s_0^\perp = 0$.

Perturbation parameters:
$\alpha := GM/(2r_0) \sim 10^{-6}$ (Sun, $r_0 = 10^9$ m)
$\epsilon := \lambda_0/(2\pi\, r_0) \sim 10^{-11}$ ($\lambda_0 = 10$ cm)

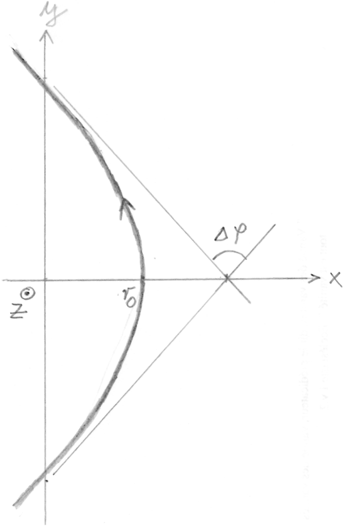

Figure: The trajectory of a photon : $\Delta\varphi$ scattering angle

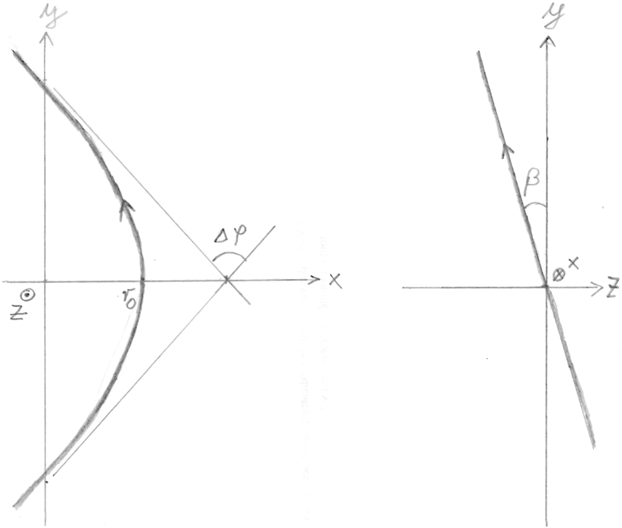

Figure: The trajectory of a photon with positive helicity: $\Delta\varphi$ scattering angle, β angle out of geodesic plane

!! The projection of the photon on the geodesic plan moves with the speed of light c.

- Scattering angle of light $\Delta\varphi \sim 4GM/r_0 + O(\alpha^2, \epsilon^2) \sim 1$ arc”
- Angle out of geodesic plane $\beta \sim -\chi\epsilon + O(\epsilon\alpha, \alpha^2, \epsilon^2) \sim 10^{-6}$ arc”
- Rainbow effect: β depends on λ_0.
- β does not depend on the mass of the Sun M.

$$\textbf{!!} \quad \vec{s}^{\perp}(t) \sim -\chi\hbar \begin{pmatrix} t/r_0 \\ 0 \\ 0 \end{pmatrix} + O(\epsilon\alpha, \alpha^2, \epsilon^2), \quad |\vec{s}^{\perp}(t)| \to \infty$$

Harvit et al. 1974: Very Long Baseline Interferometry

$$|\beta| < 10^{-3} \text{ arc”}.$$

Birefringence in linearized gravitational wave

$$d\tau^2 = dt^2 - (1 - \sigma \cos[\omega(t - z)])\, dx^2$$

$$- (1 + \sigma \cos[\omega(t - z)])\, dy^2 - dz^2 \; + \; O(\sigma^2)$$

LIGO/Virgo: $\omega = 2\pi/\lambda_{\text{GW}} \sim 2\pi \cdot 100$ Hz

$\sigma \sim 10^{-20}$

$\lambda_{\text{photon}} = 1\,\mu\text{m}$

Marsot 2019:

$$\Delta\tau_{\text{photon}}\Big(1 \qquad + \; \sigma \qquad + \sigma\, \frac{\lambda^2_{\text{photon}}}{\lambda^2_{\text{GW}}}\Big)$$

no GW	GW	GW
	no spin	spin
		$\sim 10^{-45} \ll \sigma^2$