Topolgical field Theories

Introduction

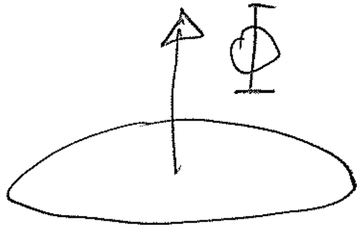
Topological concepts in physics ⟶ Nobel prize 2016

- xY model (Kosterlitz-Thouless Transition)
- quantum spin chains (Haldane's conjecture)
- quantum Hall (integer, fractional)

Topological term: operator that affects the low-energy behaviour of a theory solely on the basis of the topology of its fields

Setting the scene : a very simple example

particle of charge e on a 1D ring



minimal coupling

$$H = \frac{\hbar^2}{2mR^2}\left(-i\partial_\varphi - \frac{\phi}{\phi_0}\right)^2$$

Φ : magnetic flux , $\Phi_0 = \frac{h}{e}$ flux quantum

For simplicity : $m=1$, $R=1$; $\hbar = 1$, $e = 1 \rightarrow \Phi_0 = 2\pi$

$$H = \frac{1}{2}\left(-i\partial_\varphi - \frac{\Phi}{\Phi_0}\right)^2 = \frac{1}{2}\left(-i\partial_\varphi - A\right)^2$$

Solve exactly

$$H\psi = \mathcal{E}\psi \qquad \text{with} \quad \psi(0) = \psi(2\pi)$$

$$\psi_n = \frac{1}{\sqrt{2\pi}} e^{in\varphi} \qquad \text{and} \quad \mathcal{E}_n = \frac{1}{2}(n-A)^2$$

Let us reformulate the problem in the language of imaginary-time path integral

$$Z = \int\limits_{\varphi(\beta)-\varphi(0)\in 2\pi\mathbb{Z}} D\varphi \; e^{-\int_0^\beta d\tau \, L_E(\varphi,\dot{\varphi})}$$

- obtain $L_E(\varphi, \dot{\varphi})$

$$L_E = \frac{1}{2}\dot{\varphi}^2 - iA\dot{\varphi} \quad ; \quad S[\varphi] = \int_0^\beta d\tau \, L_E(\varphi, \dot{\varphi})$$

saddle-point analysis

$$\frac{\delta S[\varphi]}{\delta \varphi(\tau)} = 0 \quad \Longleftrightarrow \quad \frac{d}{d\tau}\frac{\partial L_E}{\partial \dot{\varphi}} - \frac{\partial L_E}{\partial \varphi} = 0$$

Euler-Lagrange equations

$$\Rightarrow \boxed{\ddot{\varphi} = 0}$$

- The vector potential does not enter the classical EOM
- Family of solutions of the form $\varphi = \alpha \tau$ with

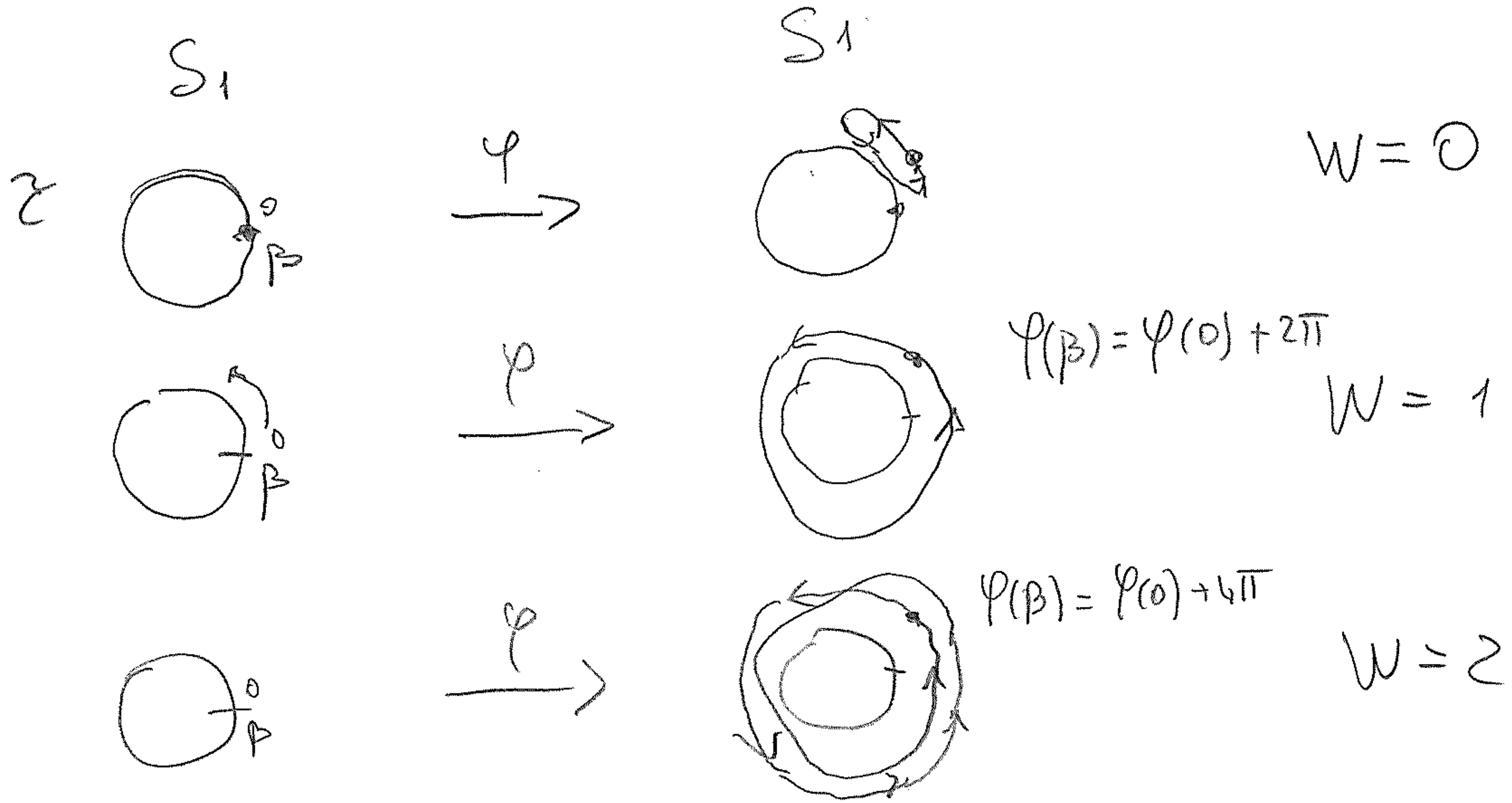
$$\varphi(\beta) = \varphi(0) + 2\pi \underset{\in \mathbb{Z}}{W} \quad \rightarrow \quad \alpha = \frac{2\pi W}{\beta}$$

$$\boxed{\varphi_W = \frac{2\pi W}{\beta}\tau}$$

solutions are defined by an integer index

Field φ is a mapping $S_1 \longrightarrow S_1$

$\tau \longrightarrow \varphi(\tau)$



W = winding number

Not possible to change W by a continous transformation of ϕ

→ Partition function as sum over different topological sectors (winding numbers)

$$Z = \sum_{W} \int_{\varphi(\beta)-\varphi(0)=2\pi W} \mathcal{D}\varphi \; e^{-\int_0^\beta d\tau \, L_E(\varphi,\dot\varphi)}$$

with $L_E = \frac{1}{2}\dot\varphi^2 - iA\dot\varphi$

$$S = \frac{1}{2}\int_0^\beta d\tau \, \dot\varphi^2 \underbrace{- iA\int_0^\beta \dot\varphi \, d\tau}_{S_{top}}$$

$$S_{top}[\varphi] = -iA\underbrace{\left(\varphi(\beta)-\varphi(0)\right)}_{2\pi W} = -iA\,2\pi W$$

$$\Rightarrow Z = \sum_{W} e^{iA2\pi W} \int_{\varphi(\beta)-\varphi(0)=2\pi W} D\varphi \; e^{-\frac{1}{2}\int_0^\beta d\tau \, \dot{\varphi}^2}$$

- S_{top} Topological term : $\underline{\theta \text{ Term}}$
- S_{top} invariant under $\tau \to \alpha\tau$ (change of metric of base manifold)
- S_{top} invariant under Wick rotation

  $S_{top} = -i2\pi AW$ (imaginary) both for real and imaginary time

This is a hallmark of Topological Terms !

Spin chains

$$\hat{H} = -J \sum_{i} \vec{S}_i \cdot \vec{S}_{i+1}$$

$J > 0$ Ferro

$J < 0$ AF

Experimental finding for AF spin chain

- half-integer spin: excitation (spin waves) have linear dispersion $\epsilon_k \propto v_s |k|$

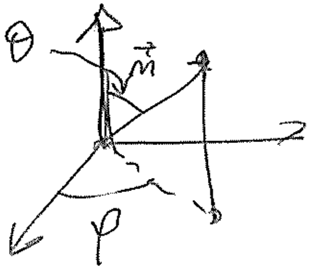
- integer-spin: excitation spectrum is gapped

$$2S \begin{cases} \text{odd}: & \text{linear dispersion} \\ \text{even}: & \text{gapped} \end{cases}$$

Is this a topological effect?

Partition function for isolated spin:

$$Z^{(1)} = \int \mathcal{D}\vec{m}\; e^{\,iS\int_0^\beta dz\, L_{wz}(\vec{m},\,\partial_z\vec{m})}$$



with $L_{wz} = (1-\cos\theta)\,\dot{\varphi}$

$$S_1 = iS\int_0^\beta dz\, L_{wz}\left(\vec{m},\,\partial_z\vec{m}\right) = iS\int_0^\beta dz\,(1-\cos\theta)\,\dot{\varphi}$$

$$\dot{\vec{m}} = \dot{\theta}\,\hat{e}_\theta + \dot{\varphi}\sin\theta\,\hat{e}_\varphi$$

$$S_1 = iS\int_0^\beta dz\;\dot{\vec{m}}\cdot\vec{A} = iS\oint_\gamma \vec{m}\cdot\vec{A} \underset{\substack{\uparrow\\ \text{Stokes}\\ \text{Theorem}}}{=} iS\int_\Gamma \underset{\substack{\uparrow\\ \text{area}\\ \text{element}}}{d\vec{S}}\,(\vec{\nabla}\times\vec{A})$$

$$\vec{A} = \frac{1-\cos\theta}{\sin\theta}\,\hat{e}_\varphi$$

$$= i S \int_\Gamma d\vec{s} \cdot \hat{e}_r = i S \Gamma[\vec{n}]$$

Excercise: action for the ferromagnetic chain

$$- J \vec{S}_i \cdot \vec{S}_{i+1} \longrightarrow - J S^2 \vec{m}_i \cdot \vec{m}_{i+1} \longrightarrow + J \frac{S^2}{2} \left(\vec{m}_i - \vec{m}_{i+1} \right)^2$$

$$\vec{S}_i = S \vec{m}_i$$

$$S[\vec{n}] = \int d\tau \sum_j \left[\frac{J S^2}{2} \left(\vec{m}_j - \vec{m}_{j+1} \right)^2 + i S L_{WZ} \left(\vec{m}_j , \partial_\tau \vec{m}_j \right) \right]$$

continuum limit $\sum_j \longrightarrow \frac{1}{a} \int dx$

$$S_{ferro}[\vec{n}] = \frac{1}{a} \int d\tau \, dx \left[\frac{a^2 J S^2}{2} \left(\partial_x \vec{m} \right)^2 + i S L_{WZ} \left(\vec{n}, \dot{\vec{m}} \right) \right]$$

$\vec{n}(x,\tau) \longrightarrow S^2$

No θ term!

AF spin chain

$\vec{m}'$: antiferromagnetic order-parameter field

$$\vec{m}'_i = (-1)^i \vec{m}_i$$

- guess minimal action consistent with global rotational invariance and that supports a wave-like mode

$$S_0[\vec{n}] = \frac{S}{4}\int d\tau\, dx \left(\frac{1}{v_s}(\partial_x \vec{n})^2 + v_s (\partial_\tau \vec{n})^2 \right)$$

$\uparrow$ spin-wave velocity $v_s = 2aJS$

Haldane

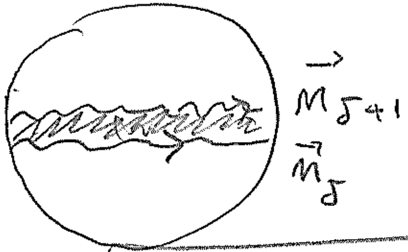
use $\vec{n}$ instead of $\vec{m}'$

Action of the <u>O(3) non-linear σ-model</u> !

Physics well known: strong quantum fluctuations induce a gap

$$S_{top}[\vec{n}] = i S \sum_{j} (-1)^{j} \int_0^\beta d\tau \, L_{WZ}(\vec{n}_j, \partial_\tau \vec{n}_j) =$$

$$= i S \sum_{j} (-1)^{j} \Gamma[\vec{n}_j]$$



$$-\left(\Gamma[\vec{n}_{j+1}] - \Gamma[\vec{n}_j]\right) \approx \int_0^\beta d\tau \; \vec{n}_j \cdot \left((\vec{n}_{j+1} - \vec{n}_j) \times \partial_\tau \vec{n}_j \right)$$

$$\boxed{S_{top}[\vec{n}] = i \theta \int d\tau \, dx \; \vec{n} \cdot \left(\partial_x \vec{n} \times \partial_\tau \vec{n} \right) \quad \text{with } \theta = \frac{S}{2}}$$

$$W = \frac{1}{4\pi} \int d^2x \; \vec{n} \cdot \left(\partial_1 \vec{n} \times \partial_2 \vec{n} \right) \in \mathbb{Z}$$

Sum over topological sectors

$$Z = \sum_{W \in \mathbb{Z}} \int \mathcal{D}\vec{n}_W \; e^{2\pi i S W} \; e^{-S_0[\vec{n}_W]}$$

$$e^{2\pi i S W} = \begin{cases} 1 & \text{if } 2S \text{ is even} \\ (-1)^W & \text{if } 2S \text{ is odd} \end{cases}$$

Topological Term is off for integer spin

→ physics of NLSM → gapped excitations

Basis of Haldane's conjecture!