TW-models for epistemic logic:

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My primary concern:

To present a kind of model (TW-models) appropriate for epistemic logic of knowledge and belief based on Timothy Williamson's conception of knowledge in his Knowledge first epistemology (*Knowledge and Its Limits*, 2000, OUP).

- In particular, I'll propose

(i) A semantic rule for the modal operator for belief based on the semantic treatment of knowledge operator in these models; and

(ii) A treatment of the problem of logical omniscience.

1. Epistemic logic—A branch modal logic

- Epistemic logic intends to provide a logical treatment of reasoning based on agents' epistemic states, typically knowledge and belief, by putting forth appropriate interpretation of some characteristic formulae in a modal language so that by taking these characteristic formulae as axioms of the established modal systems, certain logical properties of knowledge/beliefs can be characterized.
- Epistemic logic thus 'theorizes about the abstract structure of epistemic agents' theorizing'. (Williamson f.c., p.3)

1.1 A historical survey:

- 1950s: G. H. von Wright firstly presented *logical treatment of knowledge and belief*—*An Essay in Modal Logic*, London: Routledge, 1951; *Logical Studies*, London: Routledge 1957.
- 1960s: J. Hintikka equipped ***modal logics of knowledge with Kripke's possible-worlds semantic modeling of modal logics***—*Knowledge and Belief*, Ithaca, NY: Cornell University Press, 1962.
- David Lewis Present the first ***explicit definition of common knowledge among a set of agents***—*Convention: A Philosophical Study*, Cambridge, Mass.: Harvard University Press, 1969.
- Game theory (late 1960s~), together with decision theory, especially in economics.
- 1970~ non-monotonic reasoning in computer science.

1.2 Misgivings over the modal character of knowledge

- Heathcote (2004): knowledge would not have a modal character. (p.287)
- He argues that '*it is known that* . . . has the form of a modal operator may be due to superficial linguistic form and not deep logical structure'. (p.288) Based on this diagnosis his treatment is to take 'knows' as a predicate.

- But in his replies, Williamson points out that this treatment makes no obvious difference to the validity of the argument. (Williamson 2004:320)

1.3 the language for epistemic logic

- The language L_K for logic of knowledge in use:

  $p|\neg\varphi|\ \varphi\rightarrow\psi|\mathrm{K}\ \varphi$
- p: All propositional letters are formulae;
- So are $\neg\varphi$, $\varphi\rightarrow\psi$, $\mathrm{K}\varphi$
- $\mathrm{K}\varphi$: The agent knows that φ.

- Similarly. the language L_B for logic of belief in use:

  $p|\neg\varphi|\ \varphi\rightarrow\psi|\mathrm{B}\varphi$
- $\mathrm{B}\varphi$: The agent believes that φ;

1.4 Kripke's models

A Kripke model is a complex :

$$\mathsf{M} = \langle S, \sigma, R \rangle,$$

- S (a non-empty set of states, or possible worlds)
- σ: $(S \rightarrow (\mathsf{P} \rightarrow \{T,F\}))$, an assignment of a truth value of $\{T,F\}$ to the propositional letters of the language in use in every world.
- $R \subseteq S \times S$ (the required accessibility relation amongst all states)

1.5 Semantic rules and standard possible worlds semantics for epistemic logic

- M,$w \vDash p$ iff $\varsigma(w)(p) = T$, (ς makes p true at w)
- M,$w \vDash \neg\varphi$ iff it is not the case that M,$w \vDash \varphi$
- M,$w \vDash \varphi \rightarrow \psi$ iff either M,$w \vDash \psi$ or *not* M, $w \vDash \varphi$
- M,$w \vDash$ Kφ iff for all states u in S with Rwu M ,$u \vDash \varphi$

A similar rule for the modal operator of belief:

M,$w \vDash$ Bφ iff for all states u in S with Rwu M,$u \vDash \varphi$

1.6 Proof systems for knowledge and belief

1.6.1 The underlying system: CPC—classical propositional calculus.

1.6.2 Some well-known characteristic formulae

- $(K_K)K(\varphi\rightarrow\psi)\rightarrow(K\varphi\rightarrow K\psi)$ (The Distributive law)
- $(T_K)K\varphi\rightarrow\varphi$. (The Truth Axiom; factiveness)
- (4_K) $K\varphi\rightarrow KK\varphi$, (*Positive Introspective Axiom*)
- (5_K) $\neg K\varphi\rightarrow K\neg K\varphi$ (*Negative Introspective Axiom*)
- (D_K) $K\varphi\rightarrow\neg K\neg\varphi$ ($\neg K(\varphi\wedge\neg\varphi)$, or $\neg K\bot$)
- (N_K) Form $\vdash\varphi$ one can get $\vdash K\varphi$.

For a logic of belief we have exactly the same corresponding formulae, just replace '*K*' by '*B*'.

1.6.3 The constraint of accessibility relation and validity of characteristic formulae

- R to be reflexive, i.e. $\forall w \in S$, R*ww*, the Kripke models would satisfy (*T*).
- R to be transitive, i.e. $\forall w,u,v \in S$, $Rwu \wedge Ruv \rightarrow Rwv$, the Kripke models would satisfy (*4*).
- R to be equivalence, i.e. with reflexivity, transitivity and also symmetry, the Kripke models would satisfy (*T*) *+(4)+(5*).
- R to be transitive and Euclidean ($\forall w,u,v \in S$, $Rwu \wedge Rwv \rightarrow Ruv$), but not reflexive, the Kripke models would satisfy *(K)*+ (*4*)+(*5*).
- R to be serial, i.e., $\forall x \in S \exists y Rxy$ *(no end-point included)*, the Kripke models would satisfy (*D*).

1.6.4 The family of epistemic logic for knowledge

- System K = CPC + (N_K) + (K_K)
- System T = K + (T_K)
- System S4 = T + (4_K)
- System S5 = S4 +(5_K)

Usually we have systems K, T, S4, S5 for knowledge. Halpern and Moses (1985); Moses and Shoharrt (1993), van Ditmarsch et al (2008: 11)— 'The logical system S5 is by far the most popular and accepted epistemic logic (for knowledge). Really?

1.6.5 The family of epistemic logic for belief

- Corresponding logical systems for belief can be constructed by replacing '*K*' by '*B*'.
- In general, for belief, (T_B) would not hold because belief is not factive- what someone believes may not be true.]
- But (K_B), (4_B), and (5_B) seems alright. So we may accept *K4*, or *K45* as appropriate logic for belief. (Stalnaker 1993)
- Some suggest we should take so-called *KD45* or *weak S5*, by adding D_B to *K45* yields the so-called *KD45* or *weak S5*. Some take both K45 and KD45 as appropriate for a logic of belief, e.g., Moses and Shoharrt (1993).

1.6.6 The choice of appropriate logic system

- Each characteristic formula can, under appropriate intended interpretation, signify a certain property of the very concept of knowledge/belief. For example, the factivenss of knowledge can be characterized by (T_K); also, the luminosity of knowledge-knowledge is known itself, can be characterized by (4_K).

- Amongst a family of epistemic logic, which is the most appealing one?

- The search for an appropriate modal system powerful enough to capture alleged characteristics of the philosophical notion of knowledge/belief thus becomes a rather urgent, and attractive task for epistemic logicians.

1.7 The logic of knowledge-cum-belief

- Although it is widely agreed that 'for certain application the axioms of S5 are indeed a good and useful way of modeling an agent's knowledge . . . it is often desirable to . . . extend our language to talk about knowledge *and* belief'. (Halpern and Moses (1985:475)).
- Perhaps, we should take both knowledge and belief into account when we construct an epistemic logic. E.g. Kraus & Lehmann (1988:155): 'for some applications a good system has to be able to talk about belief and knowledge'.[See also Halpern and Moses (1985), Meyer (2003)]
- That is, we can construct a logic system, the language of which is: $p|\neg\varphi|\ \varphi\rightarrow\psi|K\ \varphi|B\ \varphi$, and all modal formulae can be evaluated in the same model.

1.7.1 Three approaches to a logic of K-cum-B

(i) To construct a theory of knowledge based on some established theory of belief, e.g. *KD45*. For instance, As Meyer (2003: 195) notes, W. Lenzen in 1980 suggested that we can define knowledge in terms of 'true belief' (i.e. $K\varphi =_{def} B\varphi \wedge \varphi$) on the basis of the *KD45*-axiomatization of beliefs, and then we can get a system of knowledge, known as *S4.4* , which is the logic *S4* together with the axiom

$$\models \varphi \rightarrow (\neg K \neg K\varphi \rightarrow K\varphi).$$

(ii) Shoharrt & Moses (1993) intend to define beliefs in terms of knowledge
- 'Belief as defeasible knowledge'

- The aim: (i) To define the notion of belief as 'knowledge-relative-to assumptions', based on the intuition that belief should be viewed as defeasible knowledge; and (ii) to provide complete axiomatizations of systems of belief.

(iii) The combination of a logic of knowledge and a logic of belief

- We can construct a combinatory logic system of knowledge-cum-belief. All that is required is to put forth two distinct accessibility relations R and T for K and B, respectively in the desired Kripke models. A model is thus a complex of the form M= ⟨ S, ς, R, T⟩. It can be shown that given R, a reflexive and transitive relation, and T, a serial, transitive and Euclidean relation, M would satisfy the system $S4_K$+$KD45_B$. A similar result goes for $S5_K$+$KD45_B$. It can be further stipulated that (i) T⊆R; (ii) ∀*w*,*u*,*v*∈S, if R*wu* and T*uv*, then T*wv*. Interestingly, a model of this kind, M= ⟨W, ς, R, T⟩, satisfies
- (*KB*) $\vDash K\varphi \rightarrow B\varphi$.
- (*BKB*) $\vDash B\varphi \rightarrow KB\varphi$.
- Adding (*KB*) and (*BKB*) to $S4_K$+$KD45_B$ (correspondingly, to $S5_K$+$KD45_B$), we can get KL(S4/KD45) (correspondingly, KL (S5/KD45)), Kraus and Lehmann (1986); also Battigalli and Bonanno (1999).

2. Problems with the current epistemic logic of knowledge and belief

(i) The problem of logical omniscience

(ii) To impose constraints on epistemic agents that require them to exceed every Turing machine in computational power (Williamson (forthcoming))

3. The problem of logical omniscience

3.1 Normal systems

- A modal operator M in a modal system is called *normal* if the system contains *Necessitation*—(*N*) $\vdash \varphi \Rightarrow \vdash \mathsf{M}\, \varphi$, as a rule of inference and (*K*)— $\mathsf{M}\,(\varphi \rightarrow \psi) \rightarrow (\mathsf{M}\, \varphi \rightarrow M\, \psi)$, as an axiom.
- Correspondingly, a modal system is *normal* if its primitive modal operators are normal, such as T, S4, S5 for knowledge, and S4, K45, and KD45 for belief.

3.2 The root of logical omniscience

- In a normal epistemic logic, unrestricted applications of (*N*) and(*K*) would render a truism that if the agent knows/believes a proposition φ, and φ logically implies ψ, then the agent believes ψ as well. As Levesque (1984:198) puts it, 'at any given point (world), the set of sentences considered to be believed is closed under logical consequence'.
- Also, the agent knows/believes all valid sentences.
- However, even for a rational agent in ordinary discourse, as a resource-limited being, she may know/ believe φ and $\varphi \rightarrow \psi$, but does not know/believe ψ due to a failure of drawing any connection between φ and ψ.

3.3 Characterization of logical omniscience

- Formulae, characterizing the properties of *logical omniscience,* that an epistemic logic may have:
- LO1 $\models \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- LO2 $\models \varphi \Rightarrow \models \Box\varphi$;
- LO3 $\models (\varphi \to \psi) \Rightarrow \models \Box\varphi \to \Box\psi$;
- LO4 $\models (\varphi \leftrightarrow \psi) \Rightarrow \models (\Box\varphi \leftrightarrow \Box\psi)$;
- LO5 $\models (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$;
- LO6 $\models \Box\varphi \to \Box(\varphi \vee \psi)$;
- LO7 $\models \neg(\Box\varphi \wedge \Box\neg\varphi)$;

where $\Box$ stands for either K or B. [See Meyer (2001:191), Meyer (2003:18); van Ditmarsch et al. (2008):23]

3.4.1 Way out: (i) Levesque (1984)

Levesque (1984) drew a distinction between *explicit* beliefs—a sentence is *explicitly* believed when it is actively held to be true by an agent, *and implicit* beliefs—a sentence is *implicitly* believed when it follows from what is believed.' (p.198) Accordingly, the logic of belief in question need two modal operators: 'L*'* for implicit beliefs and '*B*' for explicit beliefs. L would behave exactly like the standard modal operator for beliefs in S4 or KD45 (on standard possible worlds semantics

He in practice constructs a situation-based semantics comprehensively enough to justify that neither of the corresponding (i)-(iv) for B holds, while leaving the corresponding (v) for B as an open question.

The criticism:

- A price that Levesque has to pay is to restrict the language in use to a non-recursive one with respect to the two modal operators L and B so that 'only regular propositional sentences (with a B or a L) can occur within the scope of these two connectives.' (P.200)
- As a matter of fact, this leads to further restriction to the application of (N_K) and (N_B) as well. Obviously, both cannot applied to a theorem of higher rank (>1) with regard to modal operators B and K. Otherwise, the rule will produce some illegitimate formulae. Perhaps we should accept Williamson's suggestion in a recent paper (Williamson, forthcoming) to adopt weak version of (N_K) where (N_K) can only apply to non-modal (or objective) formulae.
- Levesque has to appeal to a four valued semantics and also the formal system he proposed for all valid sentences needs to have an aid of relevance logic for modal operator B.

3.4.2 Way out: (ii) The syntactical approach

- Fagin et. Al. (2003) proposes a kind of model as what follows:
- M = $\langle W, \sigma \rangle$, where W, a set of worlds and σ an assignment of a truth value in $\{T, F\}$ to every sentences of the language in use in every world, i.e., σ: $W \rightarrow (L_K \rightarrow \{T, F\})$.
- Clearly, all of the characteristic formulae for logical omniscience will be invalidated on the proposed models.
- However, the price is too high to pay. As Meyer (2003) points out, this way of modelling knowledge/belief does not explain anything about knowledge/beliefs: 'it is a representation method for belief rather than true modellinig'. (Meyer 2003: 19-20)

3.4.3 Way out: (iii) Impossible worlds semantics–Rantala (1982)

- M = $\langle W, \sigma, R \rangle$,
  where W = $S \cup S^*$
  σ: $(S \rightarrow (P \rightarrow \{T, F\}) \cup (S^* \rightarrow (L_K \rightarrow \{T, F\})$.
  $R \subseteq W \times W$
- Here, *S* is the set of possible worlds as in the standard semantics; and *S** is the set of *impossible worlds*, in each of which every formula can have an arbitrary assigned truth-value.

Accordingly, the truth value of a formula in a world should be specified in what follows:

- If $w \in S$, then $M,w \vDash \varphi$, if $M,w \vDash \varphi$ in according with σ: $(S \rightarrow (P \rightarrow \{T, F\})$ and the standard possible worlds semantics,
- If $w \in S^*$, then $M,w \vDash \varphi$ if $\sigma(w)(\varphi) = T$.

3.4.4 Way out: (iv) Sieve semantics (the awareness approach)

- The underlying thought: ‘to stick to the standard possible worlds semantics and then introduce some nonstandard element by means of a special (syntax-based) function, i.e. to add into the language in use one more modal operator A so that Aφ is to mean ‘the agent is aware of φ‘
- M = ⟨W, σ, R, A⟩,

  W: a non-empty set of worlds

  σ: (W →(P→{*T*, *F*}), as in the standard

  $R \subseteq W \times W$

  A : W→ ℘ (L_B), the awareness function, assigning a set of formulae to each world so that in the given world the agent will be aware of each formula of the given set.

The semantic rules for modal formulae:

M,w⊨ Aφ, iff φ∈A(w);

M,w⊨ Bφ, iff φ∈A(w) and ∀u∈WRwu
M,u⊨φ.

Note that the aforementioned sieve semantics will invalidate (LO1)-(LO6) but not (LO7). However, if the constraint on the belief-accessibility relation is dropped off, (LO7) also becomes invalid.

4. Williamson's knowledge first epistemology (*Knowledge and Its Limits* 2000)

(i) Knowing is a state of mind
(ii) Knowing is factive
(iii) The broadness of knowing(Externalist approach)
(iv) The primeness of knowing (Knowledge first!)
(v) Take knowledge as central to our understanding of belief.
(vi) Cognitive-homeless thesis
(vii) The knowledge account of evidence—One's knowledge is just one's evidence.
(viii) The knowledge account of assertion—Assert p only if one knows that p.

4.1 Knowing is a state of mind

- For an agent S, for a given proposition p, S ***knows*** *p*

$\Leftrightarrow$ S is in a certain mental state, assuming S is in a certain case α at a given time *t*.

$\Leftrightarrow$ S is in a certain condition *C* which obtains in the assumed case α. [Note that the very condition *C* is to be signified as the content of a certain proposition *p* so that when the condition *C* obtains, and when S is in a position to know it, then S *knows p*.

Williamson's notions of 'case' and 'condition':

—A *case* is a possible total state of a system, the system consisting of an agent at a time paired with an external environment, which may of course contain other subjects .(p.52)

—A case is like a possible world, but with a distinguished subject and time: a 'centred world' in the terminology of David Lewis.' (p.52)

—A condition obtains or fails to obtain in each case. Conditions are specified by 'that' clauses.

—Thus the condition that one is happy obtains in a case α if and only if in α the agent of α is happy at the time of α. (p.52)

These indicate that Kripke's possible worlds semantics would fit Williamson's conception of knowledge.

4.2 The factiveness of knowing

—'Know' is a factive mental state operator. (p.39).

—It follows that from 'S knows that *p*' one may infer '*p*'. In symbols, K$p \rightarrow p$.

—Knowing is factive: Whether one *knows* p constitutively depends on the state of one's external environment whenever the proposition is about that environment. (TW2000: 49-50)

—Knows *p* is not conceptually prior to *p*. (p.243)

—Knowledge aims at truth. (The norm of knowledge is truth.)

4.3 The broadness of knowing (p. 52)

Towards an externalist conception of knowing/ knowledge

—A case α is ***internally like*** a case β if and only if the ***total internal physical state of the agent*** in α is exactly the same as the total internal physical state of the agent in β.

—A condition C is ***narrow*** if and only if for all cases α and β, if α is internally like β then C obtains in α if and only if C obtains in β. In other words, narrow conditions supervene on or are determined by internal physical state: no difference in whether they obtain without a difference in that state.

—A condition C is ***broad*** if and only if it is not narrow.

—A state S is ***narrow*** if and only if the condition that one is in is narrow; otherwise S is broad.

—Internalism is the claim that all purely mental states are narrow; externalism is the denial of internalism.

4.4 The primeness of knowing

—Every standard analysis of knows is incorrect: ‘[N]o analysis of the concept knows of the standard kind is correct’. (TW2000: 30)

—‘Knowing does not factorize as standard analysis require.’ (p.33)

—‘The working hypothesis should be that the concept knows cannot be analysed into more basic concepts. (p.33)

—The primeness of knowing offers a better causal explanation of the connection between action and knowledge/belief.

4.5.1 Knowledge and belief

— Although, '[a]ction is often more highly correlated with belief or with true belief than with knowledge', Williamson claims that it is 'not always'. (p.86) Instead, he insists that '**the concept *knows* is fundamental**, the primary implement of epistemological inquiry'. (p.185)

—Not to analyse knowledge in terms of justified true belief; instead, take knowledge as central to our understanding of belief.

—A *rejection of 'the programme of understanding knowledge in terms of the justification of belief'*, but also pave a way of '*understanding the justification of belief in terms of knowledge*'. (pp.185-6)

4.5.2 Knowledge as the justification of beliefs

—***To believe p is to treat p as if one knows p-that is, to treat p in ways similar to the ways in which subjects treat propositions which they know***. (pp.46-7)

—'If evidence is what justifies belief, then knowledge is what justifies belief.'(p.207)

—Suppose that knowledge, and only knowledge, justifies belief.(p.185)

—'[I]t has as good a claim to conceptual truth as the proposition that knowledge entails belief.' (p.33) This indicates that we should have as a logical truth $\mathsf{K}p \rightarrow \mathsf{B}p$.

4.6 Cognitive-homeless thesis

—Knowledge requires a margin for error such that ***one is in a position to know that a condition C obtains only if C obtains in all relevantly similar case***.

—Anti-luminosity argument: "Knowledge of the present contents of one's own mind is neither unproblematic nor prior to knowledge of other things,' (p.193) [that is, No first-person authority for self-knowledge.]

Anti-KK-principle and a rejection of 5_k

—A rejection of S4 and S5 system

(KK-principle) Kp→KKp

(If an agent knows that p then she is in a position to know that she knows that p. This is also known as (4_K).)

—'That one believes p is not a luminous condition.' (p.192) Hence, it may not hold that Bφ→KBφ; hence, it may not always hold Bφ→BBφ, known as (4_B).

—A **rejection of 5_k** —¬K*p*→K¬K*p*

According to Williamson, (5_k)-Negative Introspection axiom) and (T)—K*p*→*p* (The Truth axiom) entail (B)—¬*p*→K¬K*p* (Brouwerian axiom). But (B) would fail to some cases in any reasonable sense of knowledge. Since (T) is beyond any reasonable doubt, it must be (5_K) which is problematic.

4.7 The knowledge account of evidence

—One's knowledge is just one's evidence.

4.8 The knowledge account of assertion

—One must: Assert *p* only if one knows that *p*.

5. TW-models

5.1 The underlying thought of the proposed approach:

(i) To stick to the standard possible worlds semantics. This indicates that Kripke's models would fit the desired epistemic logic.

(ii) Based on Williamson's conception knowledge as described above. This suggests that we can have a logic of knowledge-cum-belief, wherein belief can be characterized in terms of knowledge. And

(iii) To propose a treatment of the problem of logical omniscience.

A treatment of the problem of logical omniscience

To deal with the problem of logical omniscience, I shall introduce to the proposed models a certain nonstandard element by means of a special (syntax-based) function, say $\delta : S \rightarrow \wp(L_K)$, to signify Williamson's notion of 'the agent's being in a position to know a certain proposition in a state'. Accordingly, we need to add into the language in use one more modal operator, to be denoted by I so that $I_K\varphi$ is to mean 'the agent is in a position to know φ'

5.2 The construction of TW-models

A TW-model is a complex :

$$\mathsf{M} = \langle S, \sigma, \delta, R \rangle,$$

- S (a non-empty set of states)
- σ: $(S \rightarrow (\mathsf{P} \rightarrow \{T,F\})$, an assignment of a truth value of $\{T,F\}$ to the propositional letters of the language in use in every world.
- $\delta : S \rightarrow \wp(\mathsf{L}_K)$, assigning a set of formulae of the language in use L_K to each world so that in a given world whether or not the agent is ***in a position to know*** a formula can be specified. Let us call this function *ipk*-function.
- $R \subseteq S \times S$ (the required accessibility relation amongst all states, to be specified in accordance with the axioms admitted.)

5.3 The language expanded

To signify the ipk-funtion in the intended models, we add into the language L_{KB} *a further modal operator, '*I_K*' to get an expanded language of* L_{TW}*:*

$$p|\neg\varphi|\ \varphi\rightarrow\psi|K\ \varphi|B\varphi|I_K\varphi$$

- *p*: All propositional letters are formulae;
- So are ¬φ, φ→ψ, Kφ, Bφ and Iφ.
- Kφ: The agent knows that φ.
- Bφ: The agent believes that φ;
- $I_K\varphi$: The agent is in a position to know φ.

5.4_a The semantic rules for K

- The semantic rules for modal operator are stipulated as:

  $M,s\models K\varphi$ iff $\forall t\in SRst$ $\varphi\in\delta(t)$ and $M,t\models\varphi$.

Note that the condition '$\varphi\in\delta(t)$' is to emphasize that to know φ it is not sufficient to claim that φ is true in all states accessible from the given state, but also the agent must in a position to know φ in all these states.

5.4_b The semantic rules for B

In accordance with Williamson's characterization of belief in terms of knowledge, I propose a semantic rule for B:

M,$s\vDash$Bφ iff either (i) there is some state $u \in SRsu$ and $\forall v \in SRuv \rightarrow v=u$ and M,$u\vDash$Kφ, or
(ii) M,$s\vDash$Kφ.

- The conditon (ii) makes sure that a TW-model satisfies (KB) K$\varphi\rightarrow$Bφ.
- The condition '$\forall v \in SRuv \rightarrow v=u$' in (i) indicates that sometimes we just believe something without any reason.

A heuristic explanantion

- From an epistemological point of view, when we say that the truth value of a sentence α in a state *w* is determined by the truth values of certain related formulae in some (or all) accessible states, we just take what happens in those accessible states as evidence for α. Therefore, the condition can be construed as 'Whenever φ is true in *u*, the agent just takes it as true that she knows φ in *u* (i.e. $\mathsf{M}, u \vDash \mathsf{K}\varphi$), without any further evidence (from some other accessible states). This shows that $\mathsf{M}, u \vDash \mathsf{K}\varphi$ merely results from methodological considerations, rather than from any further evidence. This is precisely what Williamson's heuristic account of belief in terms of knowledge intends to say—*To believe p is to treat p as if one knows p*.

5.4_c The semantic rules for I_K

- Since we have an extra modal operator in the language in use, we need a corresponding semantic rule for the extra operator. This can be stipulated in the following way:

  $M,s \vDash I_K\varphi$ iff $\forall t \in SRst \ \& \varphi \in \delta(t)$;

Alternatively, we my have a semantic rule for K:

$M,s \vDash K\varphi$, iff $M,s \vDash I_K\varphi$, and $\forall t \in SRst M,t \vDash \varphi$.

5.5_a Constraints on the accessibility relation

(i) In view of Williamson's rejection of KK-principle and (5_K), the accessibility relation need not be equivalent, nor transitive.

(ii) The factiveness of knowing indicates the acceptance of the truth axiom, i.e. $K\varphi \rightarrow \varphi$; hence reflexivity is required.

(iii) Given that reflexivity is required, *serial* is redundant, as every state must be accessible from itself. And in fact, (D_K) $K\varphi \rightarrow \neg K \neg \varphi$ ($\neg K(\varphi \wedge \neg \varphi)$, or $\neg K\bot$) can be satisfied in the model with reflexivity. Of course, (D_B) $B\varphi \rightarrow \neg B \neg \varphi$ may not hold.

(iv) Williamson (2000:306) suggests that if we add symmetric, the model can satisfy the Brouwersche schema (*B*)) $\varphi \rightarrow K \neg K \neg \varphi$. And we may get the system KTB.

5.5_b KTB is the right epistemic logic?

- Williamson has acknowledged that

> KTB is exactly the logic for knowledge determined by the simplest version of the margin for error considerations. The only logic features essential to the binary similarity relation are reflexivity and symmetry, accessibility in the model plays the role of similarity, and KTB is the logic determined by the constraints of reflexivity and symmetry on accessibility. (Williamson 2000: 306)

5.5$_c$ Reflexivity only—A further remark

- Williamson's suggestion to accept symmetric is understandable. This is because he takes the role of similarity as what accessibility intends to play. However, if we take accessibility as an indication of evidence for our knowledge in a given epistemic state, then symmetric would not hold in some cases.

- Perhaps, all that we need is a reflexive accessibility relation on the intended TW-models. Hence KT seems to be a better choice. But . . .

5.6 Save TW-models from the problem of logical omniscience

- TW-models invalidate K-LO1— K-LO6.
- For K-LO7, it should be compatible with the rationality of human beings.
- But then, should we reject K-LO1— K-LO6 as axiom schemata or rules of inference, or theorems?
- K-LO1 $\vdash K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi))$;
- K-LO2 $\vdash \varphi \Rightarrow \vdash K\varphi$
- K-LO3 $\vdash \varphi \rightarrow \psi \Rightarrow \vdash K\varphi \rightarrow K\psi$));
- K-LO4 $\vdash \varphi \leftrightarrow \psi \Rightarrow \vdash K\varphi \leftrightarrow K\psi$;
- K-LO5 $\vdash (K\varphi \wedge K\psi) \rightarrow K(\varphi \wedge \psi)$;
- K-LO6 $\vdash (K\varphi \rightarrow K(\varphi \vee \psi)$

If so, what remains would be something too trivial to keep.

5.6 Save TW-models from the problem of logical omniscience

To Save TW-models from the problem of logical omniscience while retaining these axioms, rule of inference and theorems, we may modify the corresponding formulae as what follows:

- K-LO1 $\vdash K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow (I_K\psi \rightarrow K\psi))$;
- K-LO2 $\vdash \varphi \Rightarrow \vdash I_K\varphi \rightarrow K\varphi$
- K-LO3 $\vdash \varphi \rightarrow \psi \Rightarrow \vdash K\varphi \rightarrow (I_K\psi \rightarrow K\psi))$;
- K-LO4 $\vdash \varphi \leftrightarrow \psi \Rightarrow \vdash (I_K\varphi \rightarrow K\varphi) \leftrightarrow (I_K\psi \rightarrow K\psi))$;
- K-LO5 $\vdash (K\varphi \wedge K\psi) \rightarrow (I_K(\varphi \wedge \psi) \rightarrow K(\varphi \wedge \psi))$;
- K-LO6 $\vdash (K\varphi \rightarrow (I_K (\varphi \vee \psi) \rightarrow K(\varphi \vee \psi)$

An alternative formulation:

Alternatively, we may introduce a convention by putting all related formulae of the form $I_K\varphi$ in a given formula in question to form a conjunction Φ to stand for the conjunction $\wedge I_K\varphi_i$. Then we may have the following schemata:

- K-LO1 $\vdash\Phi\rightarrow(K(\varphi\rightarrow\psi)\rightarrow(K\varphi\rightarrow K\psi))$;
- K-LO2 $\vdash\varphi \Rightarrow\vdash I_K\varphi\rightarrow K\varphi$
- K-LO3 $\vdash\varphi\rightarrow\psi \Rightarrow\vdash \Phi\rightarrow(K\varphi\rightarrow K\psi)$;
- K-LO4 $\vdash\varphi\Leftrightarrow\psi \Rightarrow\vdash \Phi\rightarrow(K\varphi\Leftrightarrow K\psi)$;
- K-LO5 $\vdash \Phi\rightarrow((K\varphi\wedge K\psi)\rightarrow K(\varphi\wedge\psi))$;
- K-LO6 $\vdash \Phi\rightarrow(K\varphi\rightarrow K(\varphi\vee \psi))$;
- K-LO7 $S5 \vdash \Phi\rightarrow \neg(K\varphi\wedge K\neg\varphi))$

A more flexible choice??

- If my proposal is acceptable, then we may have a much more flexible choice amongst the family of modal system to see which one is the best satisfactory system of the epistemic logic of knowledge-cum-belief.

- Perhaps, with the aid of *ipk*-function and the modal operator 'I_K', some expansion of S5 remains to be acceptable. But so far, it appears that KT_{KB} is the best choice for the logic of knowledge-cum-belief.

Thanks