Fractal Classes of Matroids

General Theme

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No harm in thinking of $\mathbb{F}$ as the real numbers.

Conjecture

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Equivalently: Every minor-closed class of $GF(q)$-representable matroids has a finite number of $GF(q)$-representable excluded minors.

Custard

The class of $\mathbb{F}$-representable matroids is not well-quasi-ordered.

Conjecture

Let $\mathcal{M}$ be a minor-closed class of $GF(q)$-representable matroids. Then there is a polynomial-time algorithm to decide if a $GF(q)$-representable matroid belongs to $\mathcal{M}$.

Custard

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Proof

There are only countable many algorithms and there are uncountably many minor-closed classes of $\mathbb{F}$-representable matroids.

Bad News All Round

For both $GF(q)$ and $\mathbb{F}$, it requires exponentially many rank evaluations to decide if a matroid given by a rank oracle is representable over the field.

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Good News

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We conjecture that the good news extends to all finite fields.

Ben David and Geelen's Custard

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What about the intersection of all infinite fields?

Rota's Conjecture

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Deeper Custard

Let M be an $\mathbb{F}$-representable matroid. Then there is an excluded minor for $\mathbb{F}$-representability that contains M as a minor.

How deep can the custard get?

Let $\mathbb{F}(n)$ denote the number of n-element $\mathbb{F}$-representable matroids, and $\mathbb{F}^+(n)$ denote the number of n-element excluded minors for $\mathbb{F}$-representable matroids.

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Deep Custard Conjecture

$$\lim_{n\to\infty} \frac{\mathbb{F}(n)}{\mathbb{F}(n)+\mathbb{F}^{+}(n)} = 0.$$

Let $\mathcal{M}$ be a minor-closed class of matroids. Let $\mu(n)$ be the number of n-element members of $\mathcal{M}$ and let $\mu^+(n)$ denote the number of n-element excluded minors for $\mathcal{M}$. Let

$$f(n) = \frac{\mu(n)}{\mu(n) + \mu^+(n)}.$$

Then $\mathcal{M}$ is a fractal class of matroids if $\lim_{n\to\infty} f(n)$ exists and is equal to 0.

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Geelen's Conjecture

Fractal classes of matroids do not exist.

Let $\mathcal{S}_t$ denote the class of spikes that have at most t circuit-hyperplanes together with their minors.

Theoremoid

If $t \geq 5$, then $\mathcal{S}_t$ is a fractal class of matroids.

Say that the minor-closed class $\mathcal{M}$ is smooth if $\lim_{n\to\infty} f(n)$ exists and is equal to 1.

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Conjecture

A minor-closed class of matroids is either smooth or fractal.

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Theorem
There are uncountably many smooth classes.

Let $\mathcal{M}$ be a minor-closed class of matroids, let $\mathcal{M}_1$ denote the members of $\mathcal{M}$ together with their excluded minors, $\mathcal{M}_2$ denote the members of $\mathcal{M}_1$ together with their excluded minors etc. Then $\mathcal{M}$ is recursively fractal if $\mathcal{M}_i$ is a fractal class for all i.

Conjecture

Recursively fractal classes exist.

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If $\mathbb{F}$ is an infinite field, then the $\mathbb{F}$-representable matroids are a recursively fractal class.

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Conjecture

All fractal classes are recursively fractal.

I think the following is true.

Propositionoid

If $\mathcal{M}$ is smooth, then $\mathcal{M}_i$ is smooth for all i.

Relatively Fractal Classes

Let $\mathcal{M}$ and $\mathcal{N}$ be minor-closed classes of matroids such that $\mathcal{M} \subseteq \mathcal{N}$. Then $\mathcal{M}$ is fractal relative to $\mathcal{N}$ if the excluded minors for $\mathcal{M}$ that belong to $\mathcal{N}$ dominate the members of $\mathcal{M}$.

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In other words we change our universe to a subclass of matroids.

Conjecture

There exist classes of matroids that are fractal relative to the class of real-representable matroids.

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Perhaps real-representable spikes are worth a look?

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Let $\mathcal{G}$ be a class of graphs closed under subgraphs. Let $\mu(n)$ denote the number of members of $\mathcal{G}$ with n edges and $\mu^+(n)$ denote the number of minimal forbidden subgraphs for $\mathcal{G}$ with n eges. Then $\mathcal{G}$ is subgraph fractal if blah blah blah.

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Do subgraph fractal classes exist?

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The belief is probably not.

Is this just ad hoc combinatorics?

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Standard combinatorial thinking works for matroids over finite fields.

- Pigeonhole matroids.
- Standard complexity classes.
- Imposing structure leads to good algorithms.

Structure theory is possible for matroids over infinite fields.

Conjecture

A matroid with huge branch width has one of the following as a minor: $U_{n,2n}$; the cycle matroid of a large grid; the bicycle matroid of a large grid or its dual.

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- Are there helpful width parameters for $\mathbb{F}$-representable matroids?
- Let $\mathcal{R}_\infty$ denote the class of matroids representable over all infinite fields. Is the custard just as deep for $\mathcal{R}_\infty$ as for $\mathbb{F}$?
- Would the terminology "fractal matroid" help or hinder a grant application?