On the Existence of Indiscernible Trees

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Outline

1. Indiscernible Sequence
2. Indiscernible Array
3. Indiscernible Tree
4. Applications and Examples

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- We work in a big saturated model $\mathcal{M} \models T$.

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- $\boldsymbol{a}$, $\boldsymbol{b}$, ... are (finite) tuples in $\mathcal{M}$.
- A, B, ... are small sets in $\mathcal{M}$.
- α, ... are ordinals.
- I, J are sequences of tuples in $\mathcal{M}$.
- $M \prec \mathcal{M}$.
- Formulas are denoted by φ, ψ,...

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We write

$$\mathrm{tp}(a/A) = \mathrm{tp}(b/A),$$

if there is an A-automorphism σ of $\mathcal{M}$ with $\sigma a = b$.

Usually, $\mathrm{tp}(a/A)$ is defined as the set $\{\varphi(x) \in L(A) : \mathcal{M} \models \varphi(a)\}$. By the saturation of $\mathcal{M}$, we have the above equivalence.

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What is the indiscernibility?

Indiscernible Sequence

Definition

$I = (a_i)_{i \in \alpha}$ is called an A-indiscernible sequence if whenever $i_1 < \ldots < i_n < \alpha$ and $j_1 < \ldots < j_n < \alpha$, then

$$\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/A) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/A).$$

Remark

The following are equivalent.

- $I = (a_i)_{i \in \alpha}$ *is an* A*-indiscernible sequence.*
- *For any order preserving map* $f : \alpha \to \alpha$*, there is an* A*-automorphism* σ *of* $\mathcal{M}$ *such that*

$$a_i \mapsto a_{f(i)} \ \ (\forall i \in \alpha).$$

Existence of Indiscernible Sequence

1. Let $I = (a_i)_{i \in \omega}$ be a trivial sequence (i.e. $a_i = \mathrm{const}$). Then I is an indiscernible sequence.
2. There is a non-trivial indiscernible sequence.
   - In $(\mathbb{Q}, <)$, $0, 1, 2, ...$ is an indisecernible sequence.
   - In the field $\mathbb{C}$, a transcendence basis of $\mathbb{C}/\mathbb{Q}$ is an indiscernible sequence over $\mathbb{Q}$ (in any ordering).

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Existence of Indiscernible Sequence

Let $(x_i)_{i\in\omega}$ be a sequence of variables. Let $\Gamma((x_i)_{i\in\omega})$ be a set of L-formulas (possibly with parameters).

Definition

We say that Γ has the subsequence property if there is $I = (a_i)_{i\in\omega}$ such that any subsequence J of I realizes Γ.

J of the form $(a_{f(i)})_{i\in\omega}$ (f strictly increasing) is called a subsequence.

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Examples of Γ

The following Γ's have the subsequence property.

1. $\Gamma = \{x_i \neq x_j : i < j < \omega\}$.
2. $\Gamma = \{x_i$'s are linearly independent$\}$, in a vector space.
3. $\Gamma = \{x_i < x_j : i < j < \omega\}$, in $\mathbb{R}$.
4. $\Gamma = \{x_i < x_j : i < j < \omega\} \cup \{x_i$ is a prime number $: i \in \omega\}$, in Peano Arithmetic.

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Existence of Indiscernible Sequence

Fact

If Γ has the subsequence property, then Γ is realized by an indiscernible sequence.

Proof

- Let $I = (a_i)_{i \in \omega}$ be any realization of Γ.
- Let $\varphi(x_1, ..., x_n)$ be any formula.
- Let F be an n-place function defined by:

$$F(a_{i_1}, ..., a_{i_n}) = \begin{cases} 1 & \mathcal{M} \models \varphi(a_{i_1}, ..., a_{i_n}), \\ 0 & o.w. \end{cases}$$

- By Ramsey's theorem, there is a subsequence $J = (a_{f(i)})_{i \in \omega}$ of I such that $F(a_{f(i_1)}, ..., a_{f(i_n)})$ is constant for any $i_1 < \cdots < i_n$.
- By the subsequence property, this shows the existence of φ-indiscernible sequence realizing Γ.
- Compactness yields the existence of full indiscernible sequence.

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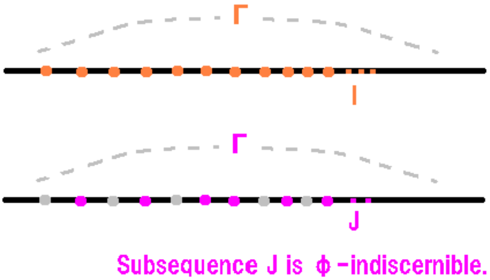
Γ
I
Γ
J
Subsequence J is ϕ -indiscernible.

Example of Arguments with Indiscernible Sequences

The following is a classical fact in model theory (T countable).

Suppose that T is ω_1-categorical. Then T is ω-stable.

T is ω-stable iff for any countable A there are at most countably many a_i's with $\mathrm{tp}(a_i) \neq \mathrm{tp}(a_j)$ $(i \neq j)$.

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Proof

- We can assume T has Skolem functions.
- Choose an indiscernible sequence I with the order type ω_1.
- Let M be the Skolem closure (definable closure) of I.
- Then $M \models T$ and $|M| = \omega_1$.
- By the indiscernibility of I, M has the property: if $A \subset M$ is countable then $\{\mathrm{tp}(a/A) : a \in M\}$ is also countable.
- By the ω_1-categoricity, every model $N \models T$ with $|N| = \omega_1$ has this property. So, T must be ω-stable.

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Indiscernible Tree

First recall:

$I = (a_i)_{i \in \omega}$ is an indiscernible sequence $\iff$ for any $X, Y \subset \omega$,

$$otp(X) = otp(Y) \Rightarrow \text{tp}(a_X) = \text{tp}(a_Y)$$

In words, I is an indiscernible sequence if whenver X and Y have a similar shape then $\text{tp}(a_X) = \text{tp}(a_Y)$.

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Indiscernible Tree

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Tree

Definition

An ordered set $O = (O, <)$ is called a tree if every $I_a = \{b \in O : b < a\}$ is linearly ordered by $<$.

Example of Trees

1. A linearly ordered set is a tree.
2. $\mathbf{2}^{<\omega}$ (the set of all finite $\{\mathbf{0},\mathbf{1}\}$-sequences) becomes a tree by $<_{ini}$ (initial segment).
3. $\omega^{<\omega}$ (the set of all finite ω-sequences) is a tree.

We are mainly interested in $\omega^{<\omega}$ and its subtree.

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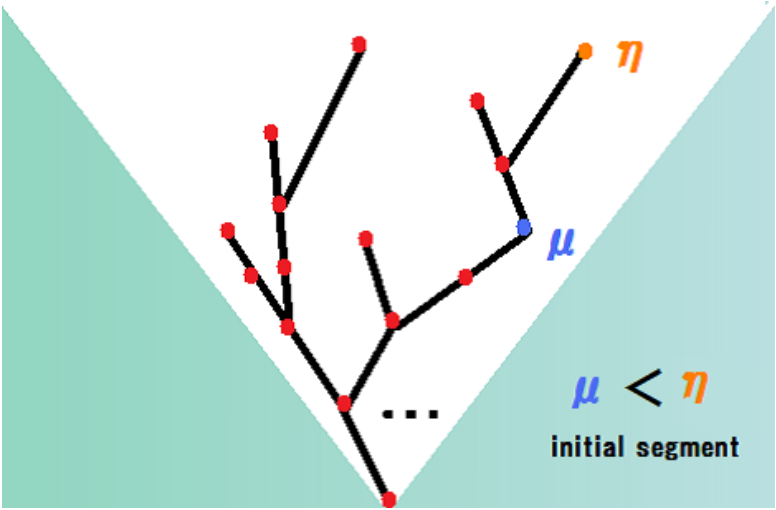
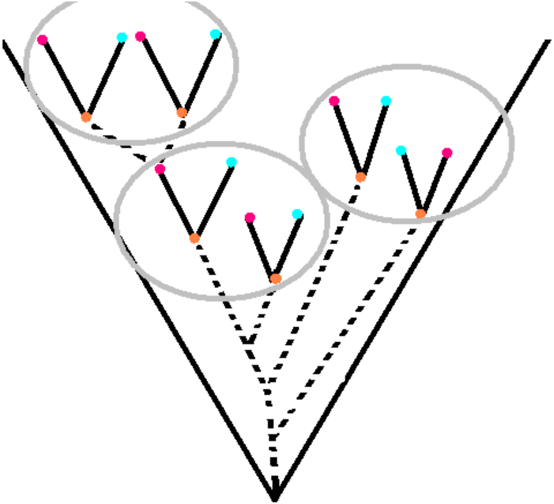
$\omega^{<\omega}$



We want to say that $I = (a_i)_{i \in O}$ is an indiscernible tree if the following condition holds:

$$X \text{ and } Y \text{ have a similar shape} \Rightarrow \mathrm{tp}(a_X) = \mathrm{tp}(a_Y).$$

Are they similar?

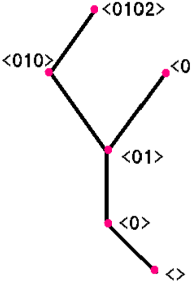


We introduce a language L_s for describing $\omega^{<\omega}$.

Definition

Let $L_s = \{<_{ini}, <_{lex}, \cap, <_{len}, P_0, P_1, P_2, ...\}$. We consider the following structure on $\omega^{<\omega}$: For $\eta, \nu \in \omega^{<\omega}$,

1. $\eta <_{ini} \nu \Leftrightarrow \eta$ is a proper initial segment of ν;
2. $\eta <_{lex} \nu \Leftrightarrow \eta$ is less than ν in the lexicographic order;
3. $\eta \cap \nu =$ the longest common initial segment of η and ν;
4. $\eta <_{len} \nu \Leftrightarrow \mathbf{len}(\eta) < \mathbf{len}(\nu)$, where $\mathbf{len}(\eta)$ is the length of the sequence η;
5. $P_n(\eta) \Leftrightarrow$ the length of η is n.



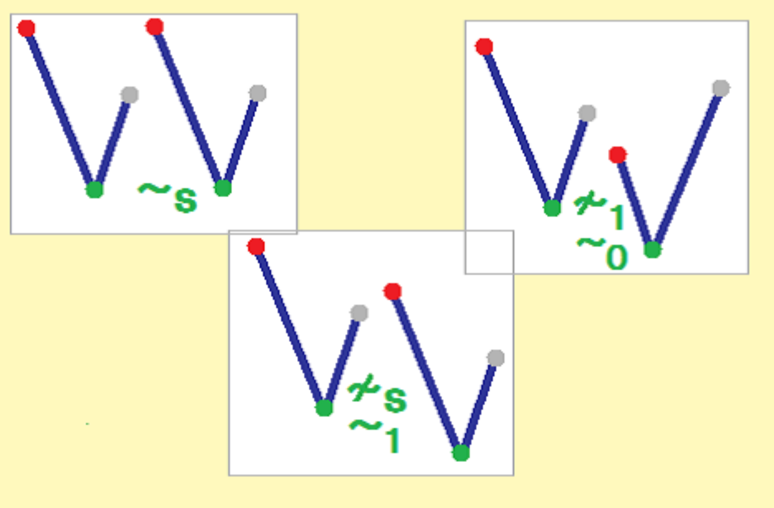
- $\langle 01 \rangle <_{ini} \langle 010 \rangle$, $\langle 0102 \rangle <_{lex} \langle 012 \rangle$.
- $\langle 010 \rangle \cap \langle 012 \rangle = \langle 01 \rangle$.
- $P_2(\langle 01 \rangle)$.
- $\langle 012 \rangle <_{len} \langle 0102 \rangle$.

Definition

1. $L_0 = \{<_{\text{lex}}, <_{\text{ini}}, \cap\}$ and $L_1 = L_0 \cup \{<_{\text{len}}\}$.
2. For $X, Y \subset \omega^{<\omega}$,
   - $X \sim_s Y \iff atp_{L_s}(X) = atp_{L_s}(Y)$.
   - $X \sim_i Y \iff atp_{L_i}(X) = atp_{L_i}(Y)$ $(i = 0, 1)$.

atp stands for 'atomic type'.

$L_0 \subset L_1 \subset L_s$



Indiscernible Tree (Weak Sense)

Definition

We say $A = (a_\eta)_{\eta \in \omega^{<\omega}}$ is a weakly indiscernible tree if

$$X \sim_s Y \Rightarrow \mathrm{tp}(a_X) = \mathrm{tp}(a_Y),$$

where $a_X = (a_\eta)_{\eta \in X}$.

This notion was introduced by Shelah. However, weak indiscernibility is our terminology.

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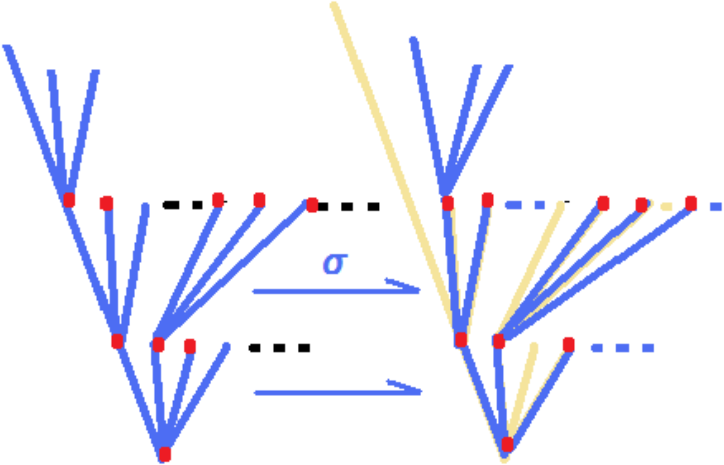
This notion was introduced by Shelah. However, weak indiscernibility is our terminology.

Definition

Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of L-formulas. We say that $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the weak subtree property if there is a realization $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ such that if $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ is an L_s-embedding then $A_\sigma = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}}$ realizes Γ.


σ

Existence of Indiscernible Trees

Theorem (Shelah)

If Γ has the weak subtree property, then Γ is realized by a weakly indiscernible tree.

The following fact is used to prove the theorem.

Fact (Shelah)

Let $O = \lambda^{<n}$ be a tree, and $f : O^k \to \mu$ a k-place function. If λ is sufficiently large (depending only on μ), then there is an L_s-embedding $\sigma : \omega^{<n} \to \lambda^{<n}$ such that $f(\sigma(X)) = f(\sigma(Y))$ for any k-tuples $X, Y \subset \omega^{<n}$ with $X \sim_s Y$.

Example of Γ having the weak subtree property

Example (Negation of Simplicity)

Suppose that T is not simple (in the sense of Shelah). Then there is $k \in \omega$, a formula $\varphi(y, x)$ and a set $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

1. $\{\varphi(y, a_{\eta|n}) : n \in \omega\}$ is consistent for each path $\eta \in \omega^\omega$ and
2. for each $\eta \in \omega^{<\omega}$ the set $\{\varphi(y, a_{\eta^\frown\langle n\rangle}) : n \in \omega\}$ is k-inconsistent.

The condition for $(a_\eta)_{\eta \in \omega^{<\omega}}$ to satisfy (1) and (2) can be expressed by a set $\Gamma((x_\eta)_{\eta \in \omega^{<\omega}})$ of L-formulas.

This Γ has the weak subtree property.

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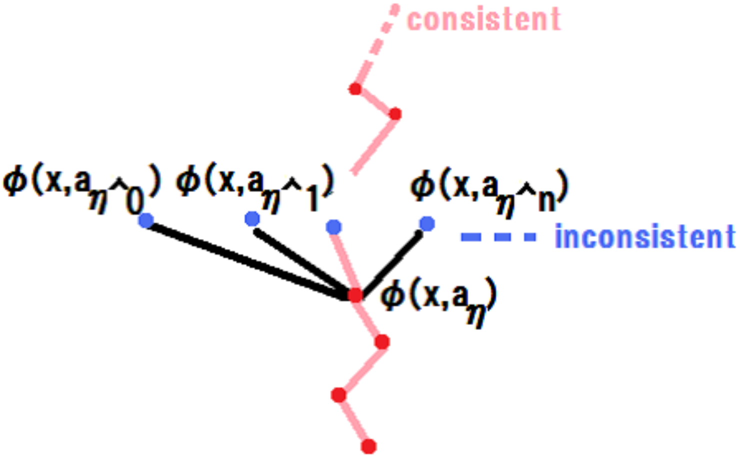
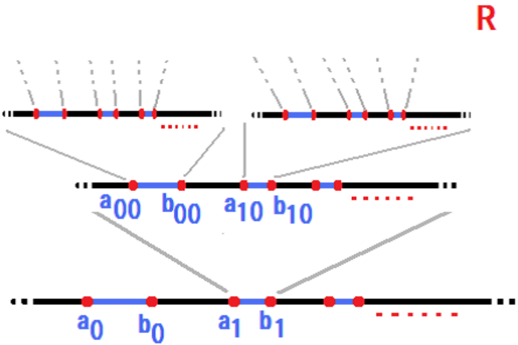

consistent
ϕ(x,a_η^0) ϕ(x,a_η^1)
ϕ(x,a_η^n)
inconsistent
ϕ(x,a_η)

Example

In the real closed field $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot, <)$, we have $A = (a_\eta, b_\eta)_{\eta \in \omega^{<\omega}}$ as shown in the following picture.



By re-choosing A from some elementary extension, A can be assumed to be a weakly indiscernible tree.

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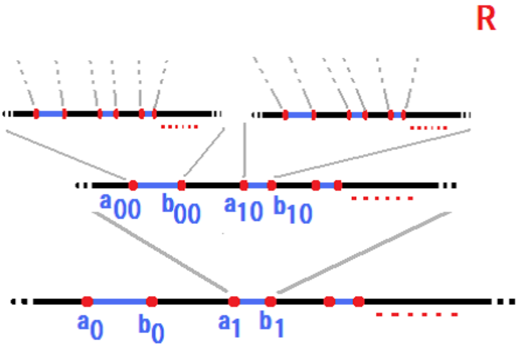

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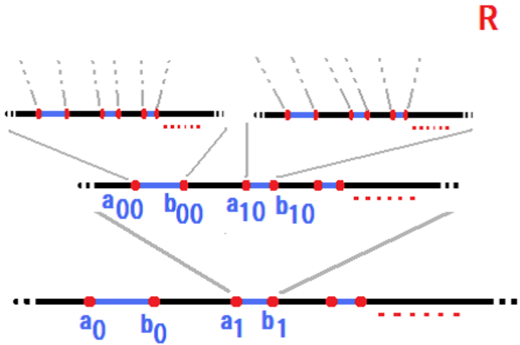

By re-choosing A from some elementary extension, A can be assumed to be a weakly indiscernible tree.

Indiscernible tree in stronger sense

Definition

$\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the subtree property
$\iff \exists A = (a_\eta)_{\eta\in\omega^{<\omega}}$ s.t.
$\forall\sigma : \omega^{<\omega} \to \omega^{<\omega}$ (L_1-embedding), $A_\sigma = (a_{\sigma(\eta)})_{\eta\in X} \models \Gamma$.

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Theorem (Takeuchi and T)

1. *If* $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ *has the subtree property, then* Γ *is realized by an indiscernible tree.*
2. *If* $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ *has the strong subtree property, then* Γ *is realized by a strongly indiscernible tree.*

Sketch of Proof

STP $\Rightarrow$ $\exists$ indiscernible tree

- Subtree Property implies Weak Subtree Property.
- So, we can choose $A \models \Gamma$, which is a weakly indiscernible tree.
- Applying Ramsey's theorem to A (+ compactness), we can choose an indiscernible tree B realizing Γ.

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- We have seen that if T is not simple then there is a set Γ expressing the nonsimplicity.
- Γ has the weak subtree property, so there is a weakly indiscernible tree realizing Γ.
- However, we sometimes want more ‘indiscernibility’ for studying this Γ.
- But this Γ does not have the subtree property nor the strong subtree property. .
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Indiscernible Trees in Other Settings

Now we consider the subtree

$$O = \{\eta \in \omega^{<\omega} : \eta(2n) = 0 \text{ for all } n \in \omega\},$$

of $\omega^{<\omega}$. We regard O as an L_s-substructure of $\omega^{<\omega}$.

We call a set $\{\eta\} \cup \{\eta^\frown\langle n\rangle : n \in \omega\} \subset O$ a family if $\eta \in O$ has odd length.

- $F(\eta_1, \eta_2) \iff \eta_1$ and η_2 belong to the same family;
- $E(\eta) \iff len(\eta)$ is even;
- $\eta_1 <_F \eta_2 \iff len(\eta_1) \leq 2n < len(\eta_2)$ for some $n \in \omega$.
- $L_{s,F} = L_s \cup \{F, E\}$,
- $L_{0,F} = L_0 \cup \{F, E\}$,
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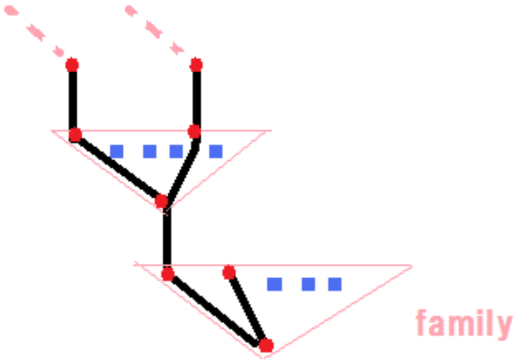

family

Definition

We say $\Gamma((x_\eta)_{\eta \in O})$ has $L_{i,F}$-subtree property if there is a realization $A \models \Gamma$ such that for every $L_{i,F}$-embedding $\sigma : O \to O$ the image A_σ realizes Γ.

Theorem (Takeuchi and T)

Suppose $\Gamma((x_\eta)_{\eta\in O})$ has $L_{i,F}$-subtree property. Γ is realized by an $L_{i,F}$-indiscernible tree.

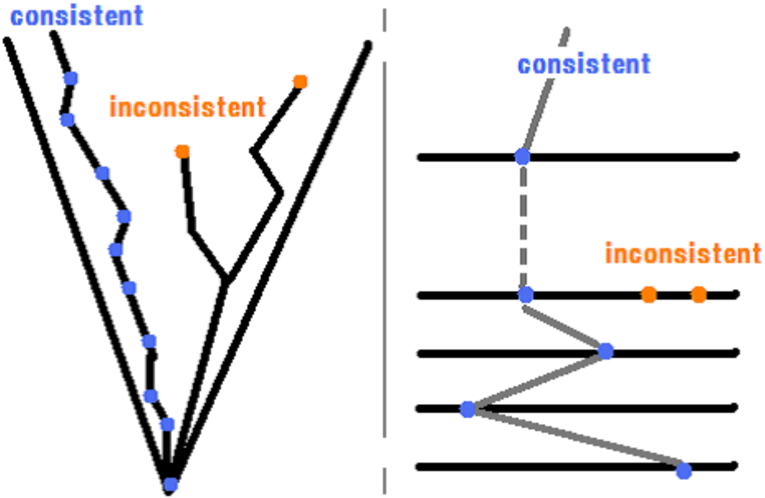
Application

Proposition (Shelah)

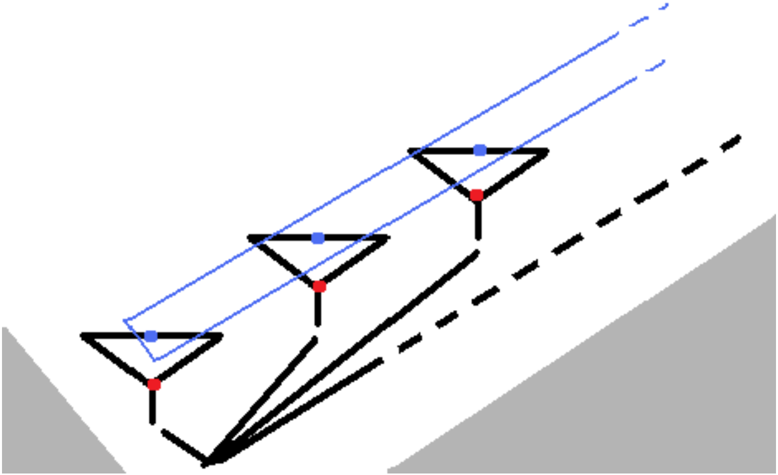
Suppose that T has the tree property. Then 1 or 2:

1. *There is a tree $C = (c_\eta)_{\eta \in \omega^{<\omega}}$ and a formula ψ such that*
    - *For each path $\eta \in \omega^\omega$, $\{\psi(x, c_{\eta|n}) : n \in \omega\}$ is consistent;*
    - *$\psi(x, c_\eta) \wedge \psi(x, c_\nu)$ is inconsistent for any incomparable η and $\nu \in \omega^{<\omega}$.*
2. *There are sets $I_i = (b_{i,j})_{j \in \omega}$ $(i \in \omega)$ with the following properties:*
    - *For each path $\eta \in \omega^\omega$, $\{\varphi(x, b_{i,\eta(i)}) : i \in \omega\}$ is consistent;*
    - *For each $i \in \omega$, $\{\varphi(x, b_{i,j}) : j \in \omega\}$ is 2-contradictory.*

Picture



Proof by Picture



Some other applications

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Thank you.