The Solovay Hierarchy

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Asian Logic Conference
December 20, 2011
Wellington, New Zealand

A phenomenon

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- The large cardinal axioms are linearly ordered according to their consistency strength
- The consistency of any mathematical theory can be reduced to that of the consistency of some large cardinal axiom.

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- Determinacy axioms such as *PD* or *AD*: used to solve problems in analysis.
- Existence of generic large cardinals: used to solve wide range of combinatorial problems.

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Remark

- *Without such reversals the phenomenon has interesting but ultimately not an important content.*
- *However, such reversals have been established for a very small initial segment of large cardinals.*

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There is no known systematic way of getting reversals much beyond the large cardinal axiom of the theorem.

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such that M is "close" to V. You are completely free to decide what "close" means here, but be careful

Theorem (Kunen)

There is no $j : V \to V$ such that $j \neq id$.

Examples

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- κ is a measurable cardinal if there is $j : V \to M$ such that $crit(j) = \kappa$ and M is closed under κ-sequence, i.e. for every $f : \kappa \to M$, $f \in M$.
- κ is a supercompact cardinal if for every λ there is $j : V \to M$ such that $crit(j) = \kappa$, $j(\kappa) > \lambda$ and M is closed under λ-sequences.

Examples

1. measurable cardinals,
2. strong cardinals,
3. Woodin cardinals,
4. Shelah cardinals,
5. superstrong cardinals,
6. subcompcact cardinals,
7. supercompact cardinals,
8. huge cardinals,
9. etc (look at Kanamori's book).

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9. etc (look at Kanamori's book).

Remark

Woodin cardinals are tiny when compared to superstrong cardinals which are tiny when compared to supercompact cardinals.

Conjecture (The PFA Conjecture)

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Remark

- *Part 3 is the most important one as it will give an equiconsistency.*
- *While the conjecture has been open for a long time, it is only a test question.*

The classical approach: the inner model problem

Problem (The Inner Model Problem)

Construct canonical inner models with large cardinals.

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Construct canonical inner models with large cardinals.

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- *The canonical inner models are models that resemble L, such models are called mice.*
- *While the problem is open for almost all large cardinals that are significantly bigger than Woodin cardinals, the desired cardinal is the supercompact cardinal.*
- *The goal is to develop tools for systematically constructing such canonical models with large cardinals while working under various theories extending ZFC.*

The origin of the problem

Definition (Gödel)

- $L_0 = \emptyset$,
- $L_{\alpha+1} = \{A \subseteq L_\alpha : A$ is definable over $(L_\alpha, \in)$ with parameters$\}$.
- for limit λ, $L_\lambda = \cup_{\alpha<\lambda} L_\alpha$.
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Theorem (Scott)

Suppose there is a measurable cardinal. Then $V \neq L$.

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- Because of this, it is impossible not to ask whether large cardinals can coexist with such a canonical structure.
- This is exactly the content of the inner model problem.
- But what are these canonical models?

The idea.

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- *An extender E is a coherent sequence of ultrafilters. It is best to think of them as just ultrafilters that code bigger embeddings than usual ultrafilters.*

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- *All large cardinals can be defined in terms of the existence of ultrafilters or extenders.*
- *An extender E is a coherent sequence of ultrafilters. It is best to think of them as just ultrafilters that code bigger embeddings than usual ultrafilters.*
- *Since all large cardinals can be defined via extenders, it is natural to look for canonical models with large cardinals among the models of the form $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders.*

The model $L[A]$

Definition (Gödel)

- $L_0[A] = \emptyset$,
- $L_{\alpha+1}[A] = \{B \subseteq L_\alpha[A] : B$ is definable over $(L_\alpha[A], \in, A \cap L_\alpha[A])$ with parameters $\}$.
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Remark

- *$L[A]$ may not be canonical, it depends on A.*
- *The idea is to consider $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders and show that it has large cardinals.*

Premice and mice

Definition

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- A mouse is an iterable premouse.
- Iterability is a fancy way of saying that all the ways of taking ultrapowers and direct limits produce well-founded models. More precisely, look at the picture.

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2. An iteration strategy for $\mathcal{M} = L_\alpha[\vec{E}]$ is a winning strategy for *II* in the iteration game on $\mathcal{M}$.
3. In general, to have a good theory of mice, $\omega_1 + 1$-iterability is all that is needed.
4. Notice that it must be hard to construct such strategies as there are trees of height ω_1 with no branch.

The inner model problem revisited.

Problem (The inner model problem)

Construct mice with large cardinals.

Mice are canonical: comparison

Definition

- Given two mice $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \trianglelefteq \mathcal{N}$ if $\mathcal{M} = L_\alpha[\vec{E}]$, $\mathcal{N} = L_\beta[\vec{F}]$, $\alpha \leq \beta$ and $\vec{E} = \vec{F} \upharpoonright \alpha$.

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- Comparison is the statement: Given two mice $\mathcal{M}$ and $\mathcal{N}$ with iteration strategies Σ and Λ, there are a Σ-iterate $\mathcal{P}$ of $\mathcal{M}$ and a Λ-iterate $\mathcal{Q}$ of $\mathcal{N}$ such that either $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$.

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Theorem (Mitchell-Steel)

Comparison holds.

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Comparison holds.

Corollary

If $\mathcal{M}$ and $\mathcal{N}$ are two mice then $\mathbb{R}^{\mathcal{M}}$ is compatible with $\mathbb{R}^{\mathcal{N}}$.

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- *There is a recent approach that goes through descriptive set theory.*
- *The classical approach, via K^c-constructions, reduces to constructing canonical iteration strategies, or $\omega_1 + 1$-iteration strategies whose ω_1 part is universally Baire. This approach, too, seems to lead to descriptive set theory.*

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- Key Point: For this to be successful, it is necessary to show that the Solovay hierarchy, just like the large cardinal hierarchy, is a consistency strength hierarchy that covers all the levels of the large cardinal hierarchy. *This has not yet been established.*

The main conjecture of descriptive inner model theory

Conjecture

The Solovay hierarchy catches up with the large cardinal hierarchy.

The Solovay sequence

Assume *AD*. First

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The Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined as follows:

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4. $\Theta = \theta_\Omega$.

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The axioms of the Solovay hierarchy are

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Theorem (Woodin)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then $AD_{\mathbb{R}}$ implies that $\Theta = \theta_\Omega$ for some limit ordinal Ω.

Some important axioms from the hierarchy

HOD is the class of hereditarily ordinal definable sets. It satisfies *ZFC*.

Examples

- $AD_{\mathbb{R}}$ + "Θ is regular".
- $AD_{\mathbb{R}}$ + "Θ is Mahlo in HOD".
- $AD_{\mathbb{R}}$ + "Θ is weakly compact in HOD".
- $AD_{\mathbb{R}}$ + "Θ is measurable".
- $AD_{\mathbb{R}}$ + "Θ is Mahlo".

More important axioms

- A set of reals is called κ-Suslin if there is a tree $T \subseteq \cup_{n<\omega}\omega^n \times \kappa^n$ such that

$$A = \{x \in \omega^\omega : \exists f \in \kappa^\omega((x, f) \text{ is a branch of } T)\}.$$

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- (LST) $AD^+ + \Theta = \theta_{\alpha+1} +$ "θ_α is the largest Suslin cardinal".
- Let ϕ be a large cardinal axiom. Then let
$$S_\phi =_{def} LST + V_\Theta^{\mathrm{HOD}} \models \exists\kappa\phi[\kappa].$$

The main conjecture of DIMT revisited

Conjecture

For each ϕ, S_ϕ is consistent relative to some large cardinal.

The consistency of the axioms.

Theorem

Suppose there is a Woodin cardinal which is a limit of Woodin cardinals. Then there is an inner model M such that $\mathbb{R} \subseteq M$ *and* $M \models AD_{\mathbb{R}} +$ "Θ *is measurable*".

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Theorem (S.-Steel)

Suppose there is a mouse with a measurable cardinal which is also a Woodin cardinal. Then there is an inner model M such that $\mathbb{R} \subseteq M$ *and* $M \models LST$*.*

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- *Proving the consistency of LST is one of the most basic open problems, but most likely it is within reach of current methods.*
- *In general, the consistency of S_ϕ for significantly strong ϕ is the most basic open problem.*
- *Tools like the core model induction have been used to establish some such consistencies.*

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Corollary

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Evidence for the DIMT conjecture

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Remark

The usual forcing methods require at least a supercompact cardinal to force either of the conclusions and both of these conclusions have a significant large cardinal strength and are probably equiconsistent with $AD_{\mathbb{R}}$ + "Θ is regular".

Forcing failure of square

Theorem (Caicedo, Larson, S., Schindler, Steel, Zeman)

Assume $AD_{\mathbb{R}}$. Suppose the set

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is stationary in Θ. Then there is a partial ordering $\mathbb{P}$ such that

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Remark

To force just $\neg\Box(\omega_2) + \neg\Box_{\omega_2}$ via conventional techniques one needs at least a subcompact cardinal which is much stronger than superstrong cardinals.

The End