A topos-theoretic approach to paraconsistent possibilistic logic

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Topoi - paraconsistency - possibilistic logic

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Logic at CLE

- Research developed under supervision of Walter Carnielli.

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- Visit www.cle.unicamp.br

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Plurality of systems: **C**-systems, LFI's, etc.

Algebra and topology

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Kind of duality: closed set topology stands for paraconsistent logic as open set topology stands for intuitionistic logic.

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Possibilistic logic expression (ϕ, α) is understood as $N(\phi) \geq \alpha$.

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Given $K = \{(\phi_i, \alpha_i)\}_{i=1,\ldots,n}$, one defines level of inconsistency
$Inc(K) = max\{\alpha | K \vdash (\bot, \alpha)\}$.

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Examples: **Set**, **Finord**, $\mathbf{Set}^{\mathbf{C}^{op}}$, $\mathbf{C}\downarrow a$.
Very interesting and useful feature: comes with logic!

Topoi and paraconsistency

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A *complement classifier* for a category E with terminal object 1 is an object Ω together with an arrow $F : 1 \to \Omega$ satisfying the condition that for every monic arrow $f : a \to b$ there exists a unique arrow $\bar{\chi}_f$ such that the following diagram is a pullback

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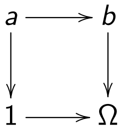

$\bar{\chi}_f$ is the complement character of f. E is a topos whose truth-values form a paraconsistent algebra.

Topoi and possibilistic logic - 1

Let $\mathcal{L}$ be a first-order language and $\mathcal{B}$ its Lindenbaum algebra. We denote by $\mathcal{M}_D$ the set of classical first-order models for $\mathcal{L}$ on the domain D.

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The set $\mathcal{P}_D$ of complete information states, equipped with the usual order, is a complete lattice.
For $L \in \mathcal{B}$, we define the *possibility of L* (according to π) as $\Pi_\pi(L) = \bigvee_{\omega \models L} \pi(\omega)$.

Topoi and possibilistic logic - 2

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$\mathbf{n}+\mathbf{1}$: if A, B and A_i are objects of $\mathbf{U}$ then the following are objects of $\mathbf{U}$: $\{1-\pi : \pi \in A\}$, $\cap A_i$, $\{\pi_1 \times \pi_2 : \pi_1 \in A, \pi_2 \in B\}$, $\{\bigvee \pi_i : \pi_i \in A_i\}$ and $\{\pi : (\forall \sigma \in A)(\pi \times \sigma \in B)\}$

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Theorem (C. Sossai)

Presheaves over **U** *can represent possibilistic logic.*

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Construct complement classifier in **U**-Sets, and investigate the obtained logic.

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Modifying Sossai's approach: at least three different choices:
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Introduce a topology in **U**, and then use closed set sheaves.
Use categorical topologies on **U** or **U**-sets.

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U. Höhle reconstructs fuzzy set theory using sheaves over complete Heyting algebras.

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Can we do something similar for possibility theory?

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Thank you!

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