A Universality Theorem

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This is a joint work with Ronald Jensen.

Part I

The Universality Theorem

I. The Universality Theorem

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Theorem (Universality)

If M is a type 1 mouse with $|M| \leq \kappa$ for some cardinal κ such that $\kappa^\omega = \kappa$ and $\kappa \geq 2^{2^{\aleph_0}}$, then the comparison of M with $\mathbb{K}^C$ terminates in fewer than κ^+ steps.

I. The Universality Theorem

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Theorem

If $\langle N_\xi \mid \xi \leq \theta \rangle$ is a K^C–array of type 1 premice such that every surviving extender on the sequence of N_ξ $(\xi \leq \theta)$ is strongly super-complete, then N_θ is a weak mouse.

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Theorem

Assume that Q is countable and normally iterable type 1 premouse. Then there is a K^C–array $\langle N_\xi \mid \xi \leq \theta \rangle$ of type 1 premice such that every surviving extender on the sequence of N_ξ $(\xi \leq \theta)$ is strongly supercomplete, and there is $\sigma : Q \rightarrow_{\Sigma^*} N_\theta$.

I. The Universality Theorem

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Definition

A premouse M is a **weak mouse** if whenever $\sigma : Q \to_{\Sigma^*} M$ and Q is countable, Q is countably iterable.

Part II

Type 1 Mice

II. Type 1 Mice

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Extenders

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A (κ, λ)–*extender* on M, ($\lambda = lh(F)$, $\kappa = crit(F)$), is a function $F : P(\kappa) \cap M \to P(\lambda)$ such that for all $v_1, \cdots, v_m < \kappa$, for all $A_1, \cdots, A_n \in P(\kappa) \cap M$, for all $B \in P(\kappa) \cap M$, if B is primitive recursive in $A_1, \cdots, A_n$ and $v_1, \cdots, v_m$, then $F(B)$ is primitive recursive in $F(A_1), \cdots, F(A_n)$ and $v_1, \cdots, v_m$ by the same definition.

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Normally we consider only *whole* (κ, λ)–extender F on M, i.e., $\lambda = F(\kappa)$.

II. Type 1 Mice

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Coherent Structures

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Coherent Structures

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(i) $E_{\omega\alpha}$ is a (κ, λ)–extender on J^E_τ and $\tau = (\kappa^+)^M < \lambda < \omega\alpha$;

(ii) $\kappa = crit(E_{\omega\alpha})$ and $\lambda = E_{\omega\alpha}(\kappa)$;

(iii) $J^E_\alpha = ult(J^E_\tau, E_{\omega\alpha})$;

(iv) κ is the largest cardinal in J^E_τ and λ is the largest cardinal in M.

II. Type 1 Mice

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Pre–pre–mice

$M = \langle J_\alpha^E, E_{\omega\alpha} \rangle$ is a *pre–pre–mouse* iff the following five conditions hold:

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Pre–pre–mice

$M = \langle J^E_\alpha, E_{\omega\alpha}\rangle$ is a *pre–pre–mouse* iff the following five conditions hold:

(a) M is acceptable.

(b) $E = \{(\nu, \eta, X) \mid \eta \leq \nu \leq \omega\alpha \ \wedge \ \eta \in E_\nu(X)\}$.

(c) If $E_\nu \neq \emptyset$, then E_ν is a whole extender on J^E_ν and $\langle J^E_\nu, E_\nu\rangle$ is coherent.

(d) If $\pi : J^E_\nu \to_{E_\nu} N$, then $E^N_\nu = \emptyset$.

(e) The restriction of M to ν, denoted by $M||\nu =_{def} \langle J^E_\nu, E_{\omega\nu}\rangle$, must be sound for all $\nu < \alpha$.

II. Type 1 Mice

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Pre–mice

A premouse is a pre–pre–mouse which satisfies an initial segment condition.

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Type 1 Pre–mice

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A pre–mouse M is *of type* 0 iff for all $\nu \leq ht(M)$ if $E^M_{\omega\nu} \neq \emptyset$ then $crit(E^M_{\omega\nu})$ is of type 0 in M.

A pre–mouse M is *of type* 1 iff for all $\nu \leq ht(M)$ if $E^M_{\omega\nu} \neq \emptyset$ then $crit(E^M_{\omega\nu})$ is of type < 2 (i.e., not of type ≥ 2) in M.

Part III

Supercomplete Extenders

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Supercomplete Extenders

Let F be an extender on $M = J^A_\alpha$. Let $\kappa = crit(F)$ and $\tau = (\kappa^+)^M$. Let $\pi : J^A_\tau \to_F J^{A'}_{\tau'}$. Let t_ξ be the ξ–th element of $J^{A'}_{\tau'}$.

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We say that F is *supercomplete on M* if and only if for every countable $X \subseteq lh(F)$, and every countable $W \subseteq P(\kappa) \cap J^A_\tau$, there is a *strong connection* $\delta : X \rightarrow \kappa$ such that

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(a) $\prec \delta(\xi_1, \cdots, \xi_n) \succ \in Z \iff \prec \xi_1, \cdots, \xi_n \succ \in F(Z)$ for $Z \in W$ and $\xi_1, \cdots, \xi_n \in X$, and

(b) if $Y \subseteq X$ and $\bigcup_{\xi \in Y} t_\xi$ is a well–founded relation, then so is $\bigcup_{\xi \in Y} t_{\delta(\xi)}$.

III. Supercomplete Extenders

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Definition

Let M be a premouse. Let $F = E_\nu^M$ be a total (κ, λ)–extender of M. Let $\alpha(F, M)$ be the largest cardinal of M in the interval $[\kappa, \lambda]$. We say that F is *strongly supercomplete with respect to* M if whenever a pair (W, X) satisfies that W is a countable set of functions and X is a countable subset of λ, there is a $g : X \to \kappa$ such that

(1) for all $h \in W$, if $dom(h) < \kappa$, then for all $x \in P(\kappa) \cap M \cap range(h)$, for all $\tilde{\alpha} \in X^{<\omega}$, $\prec \tilde{\alpha} \succ \in F(x) \iff \prec g(\tilde{\alpha}) \succ \in x$.

(2) for all $Y \subseteq X \cap \alpha(F, M)$, if $\bigcup_{\xi \in Y} t_\xi$ is well–founded, then $\bigcup_{\xi \in Y} t_{g(\xi)}$ is well–founded.

Part IV

The Model $L[\tilde{E}]$

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The Construction

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Case 1

$M_\xi = \langle J^E_\nu, \emptyset \rangle$ and there exists an F such that $\langle J^E_\nu, F \rangle$ is a premouse and F is an extender of type at most 1 and F is super complete with respect to M_ξ.

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Set $N_{\xi+1} = \langle J^E_\nu, F \rangle$.

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In any case, if $N_{\xi+1}$ is a premouse and normally iterable, then $M_{\xi+1} = core(N_{\xi+1})$; otherwise, $M_{\xi+1}$ is undefined and we stop.

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If N_λ is normally iterable, then $M_\lambda = core(N_\lambda)$.

Otherwise, M_λ is undefined and we stop.

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Theorem

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The final model is the model $L[\tilde{E}]$.

Part V

Proof Outline

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Let $\mathcal{T}_i = \langle\langle M^i_j\rangle, \langle\nu_j\rangle, \langle\pi^i_j\rangle, T^i\rangle$ $(i = 0, 1)$ be the iteration trees produced in the comparison process of length $\kappa^+ + 1$, with $M^0_0 = M$ and $M^1_0 = L[\tilde{E}]$.

V. Proof Outline

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Let θ be a sufficient large regular cardinal such that $L[\tilde{E}]_{\kappa^+}$ can be defined in H_θ and $\mathcal{T}_0$, $T^1 \in H_\theta$ and $\langle \pi^1_{0\beta} \restriction L[\tilde{E}]_{\kappa^+} \mid \beta \leq \kappa^+ \rangle \in H_\theta$.

V. Proof Outline

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Let $X_0 \prec H_\theta$ be such that $\kappa + 1 \cup \{M, \mathcal{T}_0, T^1\} \subseteq X_0$, $|X_0| = \kappa$ and $\langle \pi^1_{0\beta} \restriction L[\tilde{E}]_{\kappa^+} \mid \beta \leq \kappa^+ \rangle \in X_0$.

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Inductively define a continuous sequence of elementary substructures $X_\alpha \prec H_\theta$ of length κ^+ such that $|X_\alpha| = \kappa$ for each $\alpha < \kappa^+$. Let $\sigma_\alpha : Q_\alpha \cong X_\alpha$ be the inverse of the transitive collapse of X_α for $\alpha < \kappa^+$.

V. Proof Outline

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Let $C \subset (\kappa, \kappa^+)$ be a club such that each $\alpha \in C$ is a limit ordinal and $\alpha <_{T^i} \kappa^+$ $(i = 0, 1)$ and $\alpha = X_\alpha \cap \kappa^+$.

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Claim 1

If $\alpha \in C$, then $P(\alpha) \cap L[\tilde{E}] \subset Q_\alpha$.

V. Proof Outline

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For $\alpha \in C$ of uncountable cofinality, let E_α be the (α, κ^+)–extender on $L[\tilde{E}]$ induced by $\sigma \restriction L[\tilde{E}]$.

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Claim 2

$B = \{\alpha \in C \mid cf(\alpha) > \omega \ \&\ E_\alpha \textit{ is not supercomplete } \}$ is not stationary.

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Let $\alpha \in C$ be such that $cf(\alpha) > \omega$ and E_α is supercomplete.

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Let $\tau = (\alpha^+)^{L[\tilde{E}]}$. Let τ' and π' be such that

$$\pi' : (J^E_\tau)^{L[\tilde{E}]} \to_{E_\alpha} (J^E_{\tau'})^{L[\tilde{E}]}.$$

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$$\pi' : (J_\tau^E)^{L[\tilde{E}]} \rightarrow_{E_\alpha} (J_{\tau'}^E)^{L[\tilde{E}]}.$$

Let $\lambda \leq \kappa^+$ be least μ such that if $f \in {}^\alpha\alpha \cap L[\tilde{E}]$, and $\xi < \mu$, then $\pi'(f)(\xi) < \mu$.

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Let $F = E_\alpha \restriction \lambda$. Then F is still a supercomplete extender.

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Let $N = ult((J_\tau^E)^{L[\tilde{E}]}, F)$. Let $\pi : (J_\tau^E)^{L[\tilde{E}]} \to N$.

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Let $N = ult((J_\tau^E)^{L[\tilde{E}]}, F)$. Let $\pi : (J_\tau^E)^{L[\tilde{E}]} \to N$.

Let $\sigma : N \to (J_{\tau'}^E)^{L[\tilde{E}]}$ be the unique mapping such that $\pi' = \sigma \circ \pi$. Then σ is identity on λ, $\pi(\alpha) = \lambda$, λ is a limit cardinal in $L[\tilde{E}]$ and λ is the largest cardinal in N.

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Applying Condensation Lemma, we conclude that N is $(J_\gamma^E)^{L[\tilde{E}]}$ for some γ. This gives us a type 1 premouse $((J_\gamma^E)^{L[\tilde{E}]}, F)$.

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Since λ is a cardinal in $L[\tilde{E}]$, $(J^E_\lambda)^{L[\tilde{E}]} = M_\eta$ for some η. Since λ is the largest cardinal in $(J^E_\gamma)^{L[\tilde{E}]}$, $(J^E_\gamma)^{L[\tilde{E}]} = M_{\eta'}$ for some $\eta' > \eta$.

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By the $L[\tilde{E}]$ construction, we have that $N_{\eta'+1} = (M_{\eta'}, F)$. [By uniqueness of supercomplete extenders, which we need to prove.] Also $M_{\eta'+1} = core(N_{\eta'+1})$. This gives that $\rho_\omega(M_{\eta'+1}) < \lambda$. But, since λ is a cardinal in $L[\tilde{E}]$, if $\xi \geq \eta$, then $\rho_\omega(M_\xi) \geq \lambda$. This is our a contradiction.