# Algebraic perspective on nonclassical logics

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- Algebraic view on nonclassical logics
- In particular, algebraic aspects of syntactic properties

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  - abstract algebraic logic – W. Blok, D. Pigozzi, J. Czelakowski
  - universal algebra
  - algebraic study of substructural logics

The following 3 topics will be discussed

1. Algebra and proof theory – algebraic proof of cut elimination

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2. Relations between syntactic and algebraic properties – interpolation property and amalgamation property
3. Algebraic semantics – completions of algebras and completeness of predicate logics

♣ Though we discuss here substructural logics and residuated lattices, which are algebras for substructural logics, the arguments work well also for other nonclassical logics including modal logics.

# Substructural logics

(0) Preliminaries

♠ Let **FL** (full Lambek calculus) be the sequent system obtained from **LJ** for intuitionistic logic by deleting contraction, weakening and exchange rules and then adding rules for fusion $\cdot$.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} \ (\Rightarrow \cdot) \qquad \frac{\Sigma, \alpha, \beta, \Gamma \Rightarrow \gamma}{\Sigma, \alpha \cdot \beta, \Gamma \Rightarrow \gamma} \ (\cdot \Rightarrow)$$

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$\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ denote sequent systems obtained from **FL** by adding exchange rule, exchange and weakening rules, respectively.

♣ Substructural logics are axiomatic extensions of **FL**.

A characteristic feature of substructural logics is that implication(s) are residuals of fusion, i.e. the following four conditions are mutually equivalent in them

- $\alpha, \beta \Rightarrow \varphi$ is provable (comma)
- $\alpha \cdot \beta \Rightarrow \varphi$ is provable (fusion)
- $\beta \Rightarrow \alpha\backslash\varphi$ is provable (left-division)
- $\alpha \Rightarrow \varphi/\beta$ is provable (right-division)

# Important substructural logics

- Lambek calculus
- Linear logic
- Relevant logics
- Łukasiewicz's many-valued logics, fuzzy logics
- Johansson's minimal logic
- intuitionistic logic, classical logic, superintuitionistic logics

Substructural logics include many important nonclassical logics, and the notion is enough general to develop algebraic study in abstract way.

# Residuated lattices

An algebra $\langle L; \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is called a residuated lattice (RL) if

- $\langle L; \wedge, \vee \rangle$ is a lattice,
- $\langle L; \cdot, 1 \rangle$ is a monoid,
- $xy \leq z \;\Leftrightarrow\; x \leq z/y \;\Leftrightarrow\; y \leq x\backslash z$ for all $x, y, z$.

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An **FL**-algebra is a residuated lattice with a fixed element 0. Using 0, we can introduce two negations by defining $\sim x = x \backslash 0$ and $-x = 0/x$. $\mathcal{FL}$ denotes the variety of all **FL**-algebras.

In *commutative* residuated lattices, $x \backslash y = y / x$ holds always. In this case, residuals are denoted as $x \to y$. An $\mathbf{FL_e}$-algebra is a commutative **FL**-algebra.

Conditions $x \leq x \cdot x$, $0 \leq x$ and $x \leq 1$ correspond to contraction, right-weakening and left-weakening rules, respectively.

# Important RLs

- Lattice ordered groups: $x\backslash y = x^{-1}y,\ y/x = yx^{-1}$
- Heyting algebras: commutative residuated lattices with the least element 0 such that $x \cdot y = x \wedge y$ holds. 1 is the greatest element.
- Łukasiewicz's many-valued models:

$$x \cdot y = \max\{0, x + y - 1\}, \textit{ and } y \rightarrow z = \min\{1, 1 - y + z\}$$

- product algebra on $[0, 1]$

$$x \cdot y = x \times y, \textit{ and}$$
$$y \rightarrow z = z/y \quad \textit{if } y > z, \textit{ and } = 1 \textit{ otherwise.}$$

# Deducibility

A formula $\varphi$ is deducible from a set of formulas $\Gamma$ in a substructural logic **FL** ($\Gamma \vdash_{\mathbf{FL}} \varphi$, in symbols) when $\Rightarrow \varphi$ is provable in the calculus obtained from **FL** by adding $\Rightarrow \psi$ as an initial sequent for all $\psi \in \Gamma$. $\Gamma \vdash_{\mathbf{FL_e}} \varphi$ can be defined in the same way.

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Provability vs Deducibility

♣ [Local deduction theorem for $\mathbf{FL_e}$]

$$\Gamma, \alpha \vdash_{\mathbf{FL_e}} \varphi \textit{ iff } \Gamma \vdash_{\mathbf{FL_e}} (\alpha \wedge 1)^m \to \varphi \textit{ for some } m$$

♣ [Parametrized local deduction theorem for **FL**] – "iterated conjugates" including parameters are needed

# Algebraization theorem

A close relation between logics and classes of algebras is expressed as follows.

♣ There exists an inverse, dual lattice isomorphism between the lattice of substructural logics over **FL** and the lattice of subvarieties of $\mathcal{FL}$.

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A close relation between logics and classes of algebras is expressed as follows.

♣ There exists an inverse, dual lattice isomorphism between the lattice of substructural logics over **FL** and the lattice of subvarieties of $\mathcal{FL}$.

Moreover, the correspondence can be extended to a relation between deducibility and equational consequence.

♣ [Algebraization theorem (in the sense of Blok and Pigozzi)]
For every substructural logic **L** over **FL**, the deducibility relation $\vdash_{\mathbf{L}}$ is algebraizable, and the variety corresponding to **L** is an equivalent algebraic semantics for it.

# Algebraic cut elimination

(1) Algebra and Proof theory

♠ Gentzen's original proof gives a procedure how to eliminate each application of cut rule in a given proof. It needs double induction. Syntactic proofs of cut elimination are quite informative, as they analyze structures of proofs directly.

# Algebraic cut elimination

(1) Algebra and Proof theory

♠ Gentzen's original proof gives a procedure how to eliminate each application of cut rule in a given proof. It needs double induction. Syntactic proofs of cut elimination are quite informative, as they analyze structures of proofs directly.

Several attempts have been made to show cut elimination in an algebraic way, e.g. Maehara (1991), Okada-Terui (1996,1999) and Jipsen-Tsinakis (2002). Our proof in (BJO) is *purely algebraic*.

(BJO) F. Belardinelli, P. Jipsen and HO, Algebraic aspects of cut elimination, Studia Logica 77 (2004), 209-240.

We give an algebraic proof of cut elimination for the propositional sequent calculus $\mathbf{FL_{ew}}$, which is obtained from **LJ** by deleting contraction rule.

Cut elimination for $\mathbf{FL_{ew}}$ says:

- If a sequent $\Gamma \Rightarrow \delta$ is provable in $\mathbf{FL_{ew}}$ then it is provable in $\mathbf{FL_{ew}}$ without using cut rule.

# Completeness with respect to $\mathbf{FL_{ew}}$-algebras

Recall that an $\mathbf{FL_{ew}}$-algebras is a $\mathbf{FL}$-algebra $\mathbf{A}$ which is commutative and 0 and 1 are the least and the greatest elements.

Completeness of $\mathbf{FL_{ew}}$ with respect to $\mathbf{FL_{ew}}$-algebras.

> *A sequent* $\alpha, \beta, \ldots, \gamma \Rightarrow \delta$ *is provable in* $\mathbf{FL_{ew}}$ *iff* $\mathbf{A} \models \alpha \cdot \beta \cdots \gamma \leq \delta$ *for every* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{A}$

# Gentzen matrices

Next, we introduce algebraic structures, called Gentzen matrices for "$\mathbf{FL_{ew}}$ without cut".

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Let $B^*$ be the set of all "multisets" of members of a nonempty set $B$. $B^*$ forms a commutative monoid with respect to the multiset union, with the unit $\epsilon$, i.e. the empty multiset.

Letters $x, y, z, u, v$ $(a, b, c, d)$ are used for elements in $B^*$ (in $B \cup \{\epsilon\}$ ,resp.). $xy$ denotes the multiset union of $x$ and $y$.

A Gentzen matrix for $\mathbf{FL_{ew}}$ is a relational structure $\mathbf{B} = \langle B; \wedge, \vee, \cdot, \rightarrow; \preceq \rangle$, where $\wedge, \vee, \cdot, \rightarrow$ are binary operations on $B$, and $\preceq$ is a subset of $B^* \times (B \cup \{\epsilon\})$ such that:

- $a \preceq a$,
- $x \preceq c$ implies $dx \preceq c$,
- $x \preceq a$ and $by \preceq c$ imply $(a \rightarrow b)xy \preceq c$,
- $ax \preceq b$ implies $x \preceq a \rightarrow b$,
- $ax \preceq c$ and $bx \preceq c$ imply $(a \vee b)x \preceq c$,
- $x \preceq a$ implies $x \preceq a \vee b$,
- $x \preceq b$ implies $x \preceq a \vee b$,
- $ax \preceq c$ implies $(a \wedge b)x \preceq c$,
- $bx \preceq c$ implies $(a \wedge b)x \preceq c$
- $x \preceq a$ and $x \preceq b$ imply $x \preceq a \wedge b$,
- $abx \preceq c$ implies $(a \cdot b)x \preceq c$,
- $x \preceq a$ and $y \preceq b$ imply $xy \preceq a \cdot b$.

Each of these conditions expresses either an intial sequent, or a rule of inference (except cut) of $\mathbf{FL_{ew}}$, if $\preceq$ is replaced by $\Rightarrow$.

In $\mathbf{FL_{ew}}$-algebras, algebraic meaning of cut rule is equivalent to the following:

- $\leq$ *is transitive,*
- $a \leq b$ *implies* $a \cdot y \leq b \cdot y$.

# Completeness w.r. to Gentzen matrices

A sequent $\alpha, \beta, \ldots, \gamma \Rightarrow \delta$ is "valid" in a Gentzen matrix **B** for $\mathbf{FL_{ew}}$ iff

$$\langle g(\alpha), g(\beta), \cdots, g(\gamma) \rangle \preceq g(\delta)$$

holds in **B** for any assignment $g$.

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Since the set **Fm** of formulas (or term algebra) forms a Gentzen matrix, the following completeness of $\mathbf{FL_{ew}}$ without cut with respect to Gentzen matrices holds.

> *A sequent $\alpha, \beta, \ldots, \gamma \Rightarrow \delta$ is provable in $\mathbf{FL_{ew}}$ without cut iff it is valid in every Gentzen matrix for $\mathbf{FL_{ew}}$.*

# Towards cut elimination

Using these two completeness, we can restate cut elimination as:

- for any sequent $\Gamma \Rightarrow \delta$, if it is not valid in a Gentzen matrix $\mathbf{B}$ then it is neither valid in an $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$.

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1. There is a uniform way of constructing such $\mathbf{A}$, which is called the quasi-completion of $\mathbf{B}$.
2. In fact, $\mathbf{A}$ is a complete $\mathbf{FL_{ew}}$-algebra of the form $\mathbf{C_{B^*}}$. Here, $C$ is a nucleus on $\wp(B^*)$ and $\mathbf{C_{B^*}}$ is the algebra consisting of all $C$-closed elements.
3. Furthermore, $\mathbf{B}$ is quasi-embedded into $\mathbf{A}$.

# Quasi-completions

Each $\mathbf{FL_{ew}}$-algebra can be regarded as a Gentzen matrix satisfying the transitivity if we define $\preceq$ as follows:

$$\langle a_1, \ldots, a_m \rangle \preceq b \ \ \textit{iff} \ \ (a_1 \cdot \ldots \cdot a_m) \leq b.$$

♣ When $\mathbf{B}$ is an $\mathbf{FL_{ew}}$-algebra, the quasi-completion of $\mathbf{B}$ (as a Gentzen matrix) is isomorphic to the MacNeille completion of $\mathbf{B}$, and the quasi-embedding becomes a regular embedding.

# Cut elimination and MacNeille completion

Our algebraic proof works well not only for many standard sequent systems of substructural logics and modal logics, but also for predicate logics as long as they are complete with respect to a class of complete algebras. The idea can be applied also to completeness proofs of tableaux systems.

# Cut elimination and MacNeille completion

Our algebraic proof works well not only for many standard sequent systems of substructural logics and modal logics, but also for predicate logics as long as they are complete with respect to a class of complete algebras. The idea can be applied also to completeness proofs of tableaux systems.

From the last observation, it follows that if our algebraic proof of cut elimination works well for a sequent system of a logic **L**, then the corresponding variety must be closed under the MacNeille completion.

- Note that only three subvarieties of the variety of Heyting algebras are closed under the MacNeille completion.

# Further developments

Our argument has been developed further mainly by A. Ciabattoni, N. Galatos and K. Terui. They introduced a hierarchy of formulas, and showed results of following type.

*Let $\alpha$ belong to the class $\mathcal{N}_2$ ($\mathcal{P}'_3$). Then, $\alpha$ is preserved under (hyper)MacNeille completion iff the logic* **FL** $+ \alpha$ *admits a (hyper)sequent calculus with cut elimination.*

See also:

N. Galatos and HO, Cut elimination and strong separation for substructural logics: an algebraic approach, Annals of Pure and Applied Logic 161 (2010) 1097-1133.

# Interpolation property and amalgamation property

(2) Syntactic and Algebraic properties

♠ Next we discuss our joint work (KO) with H. Kihara. Our results can be easily extended to a much wider class of logics.

(KO) H. Kihara and H. Ono, Interpolation properties, Beth definability properties and amalgamation properties for substructural logics, Journal of Logic and Computation, 20-4 (2010), pp.823-875.

Some of related literatures are:

- J. Czelakowski and D. Pigozzi, *Amalgamation and interpolation in abstract algebraic logic*, in Lecture Notes in Pure and Applied Math. 203, 1999.
- D. Gabbay and L.L. Maksimova, *Interpolation and Definability, Modal and Intuitionistic Logics*,Oxford Logic Guides, 2005.

# Craig interpolation property

A substructural logic **L** has the Craig interpolation property (CIP) if for all formulas $\phi$ and $\psi$,

if $\vdash_{\mathbf{L}} \phi \backslash \psi$, then there exists a formula $\delta$ such that

- $\vdash_{\mathbf{L}} \phi \backslash \delta$ and $\vdash_{\mathbf{L}} \delta \backslash \psi$,
- $\mathit{Var}(\delta) \subseteq \mathit{Var}(\phi) \cap \mathit{Var}(\psi)$,

where $\mathit{Var}(\gamma)$ denotes the set of all variables in a formula $\gamma$. The formula $\delta$ is called an interpolant of $\phi \backslash \psi$.

# Deductive interpolation property

A substructural logic **L** has the strong deductive interpolation property (SDIP), if for all sets $\Gamma$ and $\Sigma$ of formulas and for all formula $\psi$,

if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$, then there exists a formula $\delta$ such that

- $\Gamma, \Sigma \vdash_{\mathbf{L}} \delta$ and $\delta \vdash_{\mathbf{L}} \psi$,
- $Var(\delta) \subseteq Var(\Gamma) \cap Var(\Sigma, \psi)$,

When $\Sigma$ is empty, we call it deductive interpolation property (DIP).

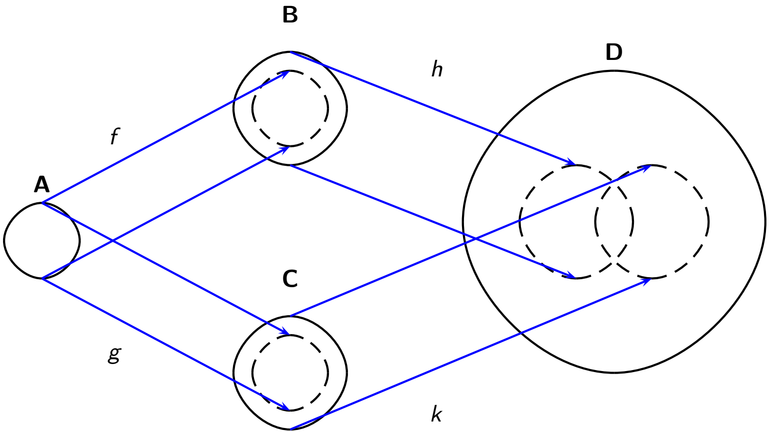
- For superintuitionistic logics, CIP $\Leftrightarrow$ DIP by using the deduction theorem,
- Only 8 superintuitionistic propositional logics have the CIP (L.L. Maksimova),
- Łukasiewicz propositional logic does not have Craig interpolation property, while it has the deductive interpolation property.

# Amalgamation property

A variety **V** of FL-algebras has the amalgamation property (AP) , if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in **V** and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$

- there exist an algebra **D** in **V** and embeddings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that
$$h \circ f = k \circ g.$$

# Amalgamation property




$$h \circ f = k \circ g$$

# Maksimova's results

To show her result, Makisimova used the following for superintuitionistic logics.

- CIP $\Leftrightarrow$ superAP $\Leftrightarrow$ AP.

Here, the superAP means AP with

> *for any $b \in B$ and any $c \in C$, if $h(b) \leq k(c)$ then there exists $a \in A$ such that $h(b) \leq hf(a) = kg(a) \leq k(c)$.*

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Also, the strongAP is defined to be the AP with

$h(B) \cap k(C) = hf(A)$.

In general,

- superAP $\Rightarrow$ strongAP $\Rightarrow$ AP.

# Robinson property

A substructural logic **L** has the Robinson property (RP), if the following holds:

Let $X$, $Y$ and $Z$ are sets of variables such that $X = Y \cap Z$, and let $Var(\Gamma) \subseteq Y$ and $Var(\Sigma) \subseteq Z$. Moreover, suppose that for each $\alpha$ such that $Var(\alpha) \subseteq X$

- $\Gamma \vdash_{\mathbf{L}} \alpha$ iff $\Sigma \vdash_{\mathbf{L}} \alpha$.

Then, for any formula $\psi$ such that $Var(\psi) \subseteq Z$,

- $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ implies $\Sigma \vdash_{\mathbf{L}} \psi$.

# RP corresponds to AP

♣ **L** has the RP if and only if the corresponding variety $V(\mathbf{L})$ has the AP (O1986 + algebraization)

For logics over $\mathbf{FL_e}$ (in fact, by local deduction theorem);

- CIP implies DIP,
- DIP is equivalent to RP,
- Hence, CIP for a logic **L** implies AP for $V(\mathbf{L})$.

# Interpolation properties for logics over $\mathbf{FL_e}$

- By the local deduction theorem for $\mathbf{FL_e}$

| | | | | |
|---|---|---|---|---|
| SCIP | $\Leftrightarrow$ | sup. RP | $\Leftrightarrow$ | CIP |
| | | $\Downarrow$ | | |
| SDIP | $\Leftrightarrow$ | RP | $\Leftrightarrow$ | DIP |

- The converse arrow does not hold – Łukasiewicz logic
- $\mathbf{L}$ has SDIP iff $V(\mathbf{L})$ has AP and congruence extension property (O1986 + algebraization).

# Interpolation properties for logics over **FL**

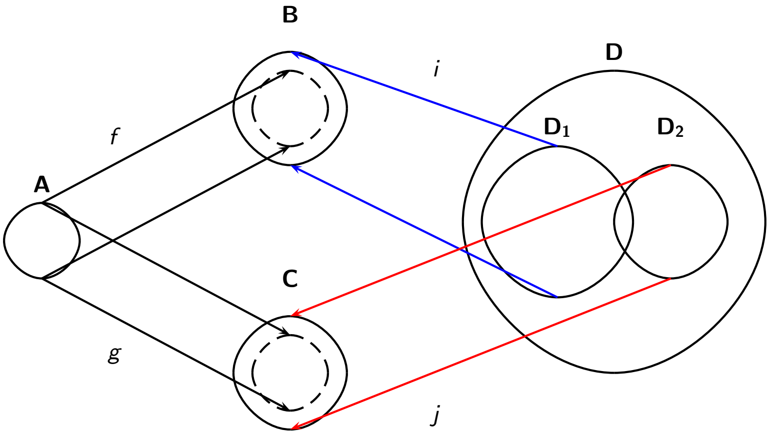
- Parameterized local deduction theorem for **FL** is not strong enough to derive the DIP from the CIP. In fact,

$$
\begin{array}{ccccc}
\text{SCIP} & \Rightarrow & \text{sup. RP} & \Rightarrow & \text{CIP} \\
 & & \Downarrow & & \\
\Downarrow & & \text{str. RP} & & \\
 & & \Downarrow & & \\
\text{SDIP} & \Rightarrow & \text{RP} & \Rightarrow & \text{DIP}
\end{array}
$$

# Algebraic characterization

| | | | | | | |
|---|---|---|---|---|---|---|
| SCIP | $\Rightarrow$ | sup. RP $\Updownarrow$ sup. AP | $\Rightarrow$ | sup. IGAP | $\Rightarrow$ | CIP $\Updownarrow$ sup. GAP |
| | | $\Downarrow$ | | $\Downarrow$ | | $\Downarrow$ |
| $\Downarrow$ | | str. RP $\Updownarrow$ str. AP | $\Rightarrow$ | str. IGAP | $\Rightarrow$ | str. GAP |
| | | $\Downarrow$ | | $\Downarrow$ | | $\Downarrow$ |
| SDIP | $\Rightarrow$ | RP $\Updownarrow$ AP | $\Rightarrow$ | DIP $\Updownarrow$ IGAP | $\Rightarrow$ | GAP |

# Generalized AP




$$\forall a \in A, \exists d \in D_1 \cap D_2 \quad (f(a) = i(d) \text{ and } g(a) = j(d))$$

1. GAP: both $i$ and $j$ are surjective homomorphisms — GAP holds for every substructural logic.
2. Injective GAP: GAP such that either $i$ or $j$ is injective — DIP
3. AP: GAP such that both $i$ and $j$ are injective — RP
4. Super GAP: GAP satisfying that for all $d_1 \in D_1, d_2 \in D_2$, if $d_1 \leq d_2$ then there exists $a \in A$ such that $i(d_1) \leq f(a)$ and $g(a) \leq j(d_2)$ — CIP

- How about interpolation properties of predicate logics?
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  - Robinson's consistency property (D. Gabbay) and inseparability (O), using Kripke semantics

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  - Maehara's proof theoretic method, using cut-free sequent calculi
  - Robinson's consistency property (D. Gabbay) and inseparability (O), using Kripke semantics
- Algebraic methods? Then, how about algebraic semantics for substructural predicate logics?

# Algebraic semantics for predicate logics

(3) Algebraic semantics for predicate logics

♠ Let us consider algebraic semantics, in which quantifiers are interpreted by infinite meets and joins (Mostowski, Rasiowa and Sikorski).

- $f(\forall x \varphi(x)) = \bigwedge \{f(\varphi(w)) : w \in V\}$
- $f(\exists x \varphi(x)) = \bigvee \{f(\varphi(w)) : w \in V\}$

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A structure $\langle \mathbf{A}, V \rangle$ is an algebraic frame for a substructural predicate logic if $\mathbf{A}$ is a complete $\mathbf{FL}$-algebra and $V$ a non-empty set, called the individual domain.

# Completions and algebraic completeness

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Since Lindenbaum algebra of a given predicate logic is not complete, we need to find a complete algebra which "reflects" the structure of Lindenbaum algebra.

A *completion* of a given **FL**-algebra **A** is a pair $(\mathbf{C}, h)$ of a complete CRL (**FL**-algebra) **C** and an embedding $h$ from **A** to **C**. Usually we call this **C** a completion of **A**.

For our purpose, $h$ must be moreover regular, i.e. preserves existing joins and meets, in order to preserve the interpretation of quantifiers.

# Algebraic incompleteness

Apparently, the idea of our algebraic frames has a limitation.

♣ **QInt** + (*strong MP*) is algebraically incomplete, where (strong MP) is: $\neg\neg\exists x\alpha(x) \rightarrow \exists x\neg\neg\alpha(x)$. (O1973, O1999)

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Many of algebraic incompleteness known up to now are caused by a difference between instantiation in algebra and substitution instance in logic.

There exist uncountably many superintuitionistic predicate logics which are neither Kripke complete nor algebraically complete.

# Algebraic completeness

Regular completions

I. MacNeille completions

Rasiowa in 1951 proved the completeness of **QInt**, by using MacNeille completion of "*Boolean algebras*".

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Regular completions

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Similarly, we can show

- algebraic completeness of basic substructural logics including $\mathbf{QFL}, \mathbf{QFL_e}, \mathbf{QFL_{ew}}$

II. Crawley completions (complete ideal completions) Crawley 1962

For a given upper semilattice **P**, a subset $J$ of $P$ is a complete ideal iff $J$ is downward closed and respects existing infinite joins, i.e., if $\bigwedge_{i \in I} a_i$ exists for $\{a_i : i \in I\} \subseteq J$ then $\bigwedge_{i \in I} a_i \in J$.

The set of all complete ideals of a given $\mathbf{FL_e}$-algebra **A** forms a complete $\mathbf{FL_e}$-algebra, which is called the Crawley completion of **A**.

Crawley completions preserve distributivity and also join infinite distributivity (JID).

- (JID) : $\bigvee_i a_i \wedge b = \bigvee_i (a_i \wedge b)$

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▶ Algebraic completeness using Crawley completions? E.g., for substructural predicate logics with the distributive law and with the axiom $(\wedge, \exists)$: $\exists x \varphi(x) \wedge \psi \to \exists x(\varphi(x) \wedge \psi)$?

▶ How about the axiom of constant domain (CD) : $\forall x(\alpha(x) \vee \beta) \to \forall x\alpha(x) \vee \beta$, which corresponds to the meet infinite distributivity (MID): $\bigwedge_i a_i \vee b = \bigwedge_i (a_i \vee b)$?

1. Kripke-completeness implies algebraic completeness for every extension of **QInt** $+$ *CD*.
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▶ How about algebraic completeness of substructural predicate logics with (CD)? Rasiowa-Sikorski lemma?

Further references to semantics for nonclassical predicate logics


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