Computational aspects of hyperimmune-free degrees

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Motivating questions

- How can we measure the “computational strength" of a real? Different ways?
- Do they (should they) have anything to do with each other?

Measuring Computational Strength

- Classically, relative computational content is measured via Turing reducibility (and variations).
  - So A is strong if it computes a large number of sets.
  - Comparing A with other complicated sets, e.g. the Halting problem $\emptyset'$.
  - Domination properties.
- Measure computational strength by dynamic properties.
  - Effective enumeration, speed of enumeration.
  - As oracle, how readily they permit numbers to enter sets they compute ($\emptyset$ =weak, $\emptyset'$ =strong).

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Computing fast growing functions

- For this talk we'll focus on first aspect, Turing reducibility. Various ways of measurement.
  1. Calibrating via jump of the set
  2. *A* able to compute a fast growing function
- Appears that they have (somewhat) to do with each other
- $A \geq_T \emptyset'$ iff A computes some f dominating every partial recursive ψ.
- (Martin) $A' \geq_T \emptyset''$ iff A computes some f dominating every total recursive ψ.

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Computing fast growing functions

- Since A is low$_2$ iff $\emptyset'$ is high relative to A, we can also look at domination in the reverse sense:
  A is low$_2$ iff there exists $f \leq_T \emptyset'$ dominating every A-recursive function.
- Similarly A is low iff there exists $f \leq_T \emptyset'$ dominating every partial-A-recursive function.
- It's also possible to define a class based on domination properties and then obtain results about it's computational properties.
- For instance array computability (Downey, Jockusch, Stob).
  A is array computable iff there exists $f \leq_{wtt} \emptyset'$ dominating every A-recursive function.
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Uniformly almost everywhere domination

- Another notion of domination: we say that A is uniformly almost everywhere dominating if there is some $f \leq_T A$ such that for almost all reals X and all $g \leq_T X$, we have f dominates g.
- This of course implies A is high, and in fact strongly resemble $\emptyset'$.
- (Kurtz) $\emptyset'$ is u.a.e.d.
- (Cholak, Greenberg and Miller) An incomplete r.e. u.a.e.d.

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Uniformly almost everywhere dominating

- (Barmpalias and Montalbán) u.a.e.d. degree which is cappable (half of a minimal pair).
- (Binns, Kjos-Hanssen, Lerman, Solomon and Simpson) Every u.a.e.d. degree is in fact superhigh.
- Perhaps the most fascinating result is of Kjos-Hanssen et al: A Δ^0_2 set A is u.a.e.d. iff $\emptyset'$ is K-trivial relative to A.
- Again this shows that a highness property can be expressed in the form of $\emptyset'$ is "low" relative to it.

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Hyperimmunefree degrees

- We focus on a fascinating class. A degree ***a*** is hyperimmunefree (HIF) if it contains no hyperimmune set. This has an equivalent characterization in terms of domination (or rather, being dominated):
- ***a*** is HIF if every function $f \leq_T A$ is dominated by a recursive function. In a way this says that HIF $=$ "almost recursive".
- So it is worth studying different computational aspects of HIF-ness.
- In particular, how being dominated relates to other notions (jump operator, relativizations).

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Hyperimmunefree degrees

- Each HIF is clearly not high, and is the opposite of being dominant.
- In fact each HIF "preserves highness", in the sense that if A is HIF and B is a high set then $(A \oplus B)' \geq A''$. No high set preserves highness.
- Each HIF is array computable, because there exists a $f \leq_{wtt} \emptyset'$ majorizing every partial recursive function.
- Each HIF is GL_2, in the sense that $A'' \leq_T (A \oplus \emptyset')'$.
- Some HIF is GL_1, i.e. $A' \leq_T A \oplus \emptyset'$.
- Strangely, the only HIF degree below $\mathbf{0}'$ is $\mathbf{0}$.

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Hyperimmunefree degrees

- On the other hand, a HIF can be DNR and even PA-complete.
- We first study HIF and the jump operator.

Question

Which degree $\mathbf{c} \geq \mathbf{0}'$ is the jump of a HIF?

- We must first understand the difficulties in coding in information into a HIF.

Proposition

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- Let's analyze this. Recall the standard construction of a HIF.
- Force with total recursive perfect trees, $T_0 \supset T_1 \supseteq T_2 \supseteq \cdots$, where for each i we force Φ_i^X to be totally convergent, or totally divergent on T_i.
- The coding location of $C(i)$ is at the first split of T_i.
- The entire construction is recursive in $C \oplus \emptyset''$.

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- Now on each T_i we can ask $\emptyset''$ if Φ^X_{i+1} is total or not. Hence $A'' = TOT^A \leq_T C \oplus \emptyset''$.
- If we are forcing $\Sigma^0_1(A)$ predicates, this can be recovered in $C \oplus \emptyset'$.
- The complicated part is the coding of C. Coding location of C recoverable only in $A \oplus \emptyset''$.
  $\Rightarrow$ We do not have jump inversion for HIF.
- We don't even have $A' \geq_T C$ for every (any) $C \geq_T \emptyset''$!

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Coding C into A'

- To understand the difficulty of coding information into (the jump of) a HIF, let's study a weaker property.

Proposition (Barmpalias, Downey, N)

If $\mathbf{c} \geq \mathbf{0}''$ then there is $\mathbf{a} > \mathbf{0}$ with $\mathbf{a}' = \mathbf{c}$, and $\mathbf{a} \cap \mathbf{0}' = \mathbf{0}$.

- Use $\emptyset'$ to find the first place η_0, η_1 (if it exists) where $\Phi_e^{\eta_i}$ splits. If not found then we're done.
- Code $C(e)$ above this.
- Crucial point: Even if paths $X \supset \eta_i{}^\frown 1$ diverges, we can still make $A \supset \eta_i{}^\frown 0$. This does no harm to the minimal pair requirement.

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- In fact, for the weaker property of "forms minimal pair with $\emptyset'$" we have

Theorem (Barmpalias, Downey, N)

A degree $\mathbf{c} \geq \mathbf{0}'$ is hyperimmune relative to $\mathbf{0}'$ iff it is the jump of a weakly 2-random not 2-random.

- Attempting to do the same thing for HIF sets fails. The main difficulty is that we may first set σ_0, σ_1 to be the coding location for some $C(n)$.
- Much later on we may discover that paths $X \supset \sigma_1$ may diverge. This will usually take r.e. relative to $\emptyset'$ to find.
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The jump of non-zero HIF degrees

- The difficulty in coding has meant that more negative results have been found.
- (Jockusch, Stephan) If A' is *PA* relative to $\emptyset'$ then A is cohesive hence of hyperimmune degree.
- (Kučera, Nies) If $C >_T \emptyset'$ is Σ^0_2 then there is a non-recursive HIF A such that $A' \leq_T C$.
- Question: Is every (or any) Σ^0_2 degree $\mathbf{0}' < \boldsymbol{c} < \mathbf{0}''$ the jump of a HIF?

The jump of non-zero HIF degrees

- Surprisingly,

Theorem (NSYY)

If A is HIF then A' cannot compute any Σ^0_2 degree other than $\mathbf{0}'$.

Sketch of proof.

- Suppose $\lim_s f^A(x,s) = 1$ iff $(\exists s)(\forall t > s)R(x,t)$.
- Let $g(x,s)$ to be the first $t > s$ found such that $\neg R(x,t)$ or $f(x,t) = 1$.
- Since g is majorized by a recursive function, can use this to get a bound for each s the stage $t > s$ such that $\neg R(x,t)$.
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Jumps of HIFs cannot be HIF($\emptyset'$)

- So no jump can be $PA(\emptyset')$ nor compute a Σ^0_2 degree.
- Surprisingly the jump of a HIF cannot be HIF($\emptyset'$)!

Lemma

If a tree $T \leq_T \emptyset'$ contains no left r.e. path, then no path of T is HIF.

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- Let $T = \lim_s T_s$ limit of recursive trees.
- If T contains some HIF path A, then let $f^A(n)$ be the first stage s found such that $A \upharpoonright n \in T_s$.
- Can use a recursive majorant of f^A to define a Π^0_1 class $P \subseteq [T]$ containing A.
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Theorem (NSYY)

If A is HIF and $A \oplus \emptyset'$ is HIF($\emptyset'$) then A is recursive.

Sketch of proof.

- The idea is to show that we can start with 2^ω and remove all left r.e. paths $\alpha_0, \alpha_1, \cdots$ without removing A.
- If we do this with a $\emptyset'$-recursive tree, then we get a contradiction by previous lemma.
- Let $f^{A\oplus\emptyset'}(n) = \min\{m \mid A \upharpoonright m \not\subset \alpha_i \text{ for any } i \leq n\}$ be dominated by $g \leq_T \emptyset'$.
- Now remove $\sigma \in 2^{g(n)}$ iff one of $\alpha_0, \cdots, \alpha_n \supset \sigma$. □

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Every 2-generic is the jump of a HIF

Theorem (NSYY)

If C is 2-generic then $C \oplus \emptyset' \equiv_T A'$ for some HIF A.

Sketch of proof.

- Recall the difficulty of coding C into HIF-trees: Some divergence is found only after a coding location is set up.
- These are Σ^0_2 events, so if C is 2-generic, then C will want to go there anyway. □

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  4. Contains $C \oplus \emptyset'$ for every 2-generic C.

Question

- *Is the jump of every HIF the jump of a recursively traceable?*
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HIF relativized

- We now turn to studying relativization of HIF.
- We say that A is HIF(B) if every function recursive in $A \oplus B$ is dominated by a B-recursive function.
- Question: To what extent can a HIF set be HIF relative to other sets?
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There exists a HIF set A such that A is HIF relative to every low c.e. set.

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For which other c.e. set B is it possible for a HIF set to be HIF(B)?

- Problem: No incomplete c.e. set bounds every left r.e. real / superlow set.
- How well does HIF interact with being random?
  By the HIF basis theorem, there exists HIF random sets.

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- Here A is a notion of lowness.

Proposition (SNYY)

Let $A \leq_T \emptyset'$. Then A is K-trivial iff some HIF set is A-random.

Proof.

- Suppose A is not K-trivial, and B is a HIF A-random.
- There is a $\Pi^0_1(A)$-class containing B and only A-random sets.
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HIF random sets

- Generalizing this globally to low for Ω sets fails: Take A HIF and a set B which is A-random and A-HIF.
- We replace A-random by A-DNR, and ask the same question.

Theorem (SNYY)

Let $A \leq_T \emptyset'$.

- *Suppose some HIF set A-tt computes an A-DNR function, then A is low.*
- *If A is a low r.e. set, then some HIF set is A-DNR.*

- Are the low sets exactly the class of sets A below $\emptyset'$ where some HIF is A-DNR?

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HIF and closed sets

- HIF is intimately related to closed sets
- (Jockusch, Soare) HIF Basis Theorem
- Standard construction of a HIF involves forcing with total recursive trees.
- Recursive trees are the only way we know how to directly construct HIF.
- Interactions of HIF and trees.
- Miller and Nies observed that no real can be simultaneously HIF, DNR and GL_1, although any two combination are possible.
- To what extent can these be demonstrated in a Π^0_1 class? (E.g. Π^0_1 class of DNR, Π^0_1 class of GL_1).

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HIF and DNR

Proposition (SNYY)

If P is a Π^0_1 class containing only HIFs. Then P cannot contain a DNR.

Proof.

- Suppose P contains only HIFs, and some DNR A.
- Since A is HIF, there is some total functional Ψ such that Ψ^A is DNR.
- The set $Q = \{X \in 2^\omega \mid \Psi^X \text{ is not DNR}\}$ is open.
- $P - Q$ is a Π^0_1 class containing only HIFs and no non-recursive path. Contradiction. □

HIF and DNR

Proposition (SNYY)

If P is a Π^0_1 class containing only HIFs. Then P cannot contain a DNR.

Proof.

- Suppose P contains only HIFs, and some DNR A.
- Since A is HIF, there is some total functional Ψ such that Ψ^A is DNR.
- The set $Q = \{X \in 2^\omega \mid \Psi^X \text{ is not DNR}\}$ is open.
- $P - Q$ is a Π^0_1 class containing only HIFs and no non-recursive path. Contradiction. □

DNR and GL_1

Proposition (SNYY)

There is an uncountable Π^0_1 class P such that every non-recursive path of P is GL_1 and DNR.

Proof.

- Let $T \leq_T \emptyset'$ be a tree containing only 2-random reals.
- Build a recursive tree Q such that every non-isolated path of Q is Turing equivalent to a path of $[T]$, and vice versa. □

HIF and GL_1

The (apparently) most difficult combination:

Theorem (SNYY)

There is an uncountable rank 1 Π^0_1 class P such that every non-recursive path of P is HIF and GL_1.

Proof.

- The proof involves building *P* and fully approximating the HIF construction on *P*.
- A tree of strategies control the different versions of HIF requirements.
- Paths are isolated when some HIF strategy hits a Σ_2 outcome, otherwise it defines a total recursive subtree.
- To combine with GL_1 we must ensure that we do not need to know the True Path of construction. □

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- Thank you.