Promptness, Randomness, Capping and Cupping

Anthony Morphett

Asian Logic Conference
Wellington
19 December 2011

Prompt enumerations and permitting

Important notion from study of c.e. sets & degrees:

prompt permitting (prompt simplicity)

A c.e. set A is promptly permitting if there is a computable function p such that, for all c.e. sets W,

$$|W| = \infty \Rightarrow \exists^\infty x, s : x \in W[\text{at } s] \text{ and } A[s] \upharpoonright x \neq A[p(s)] \upharpoonright x.$$

The **promptly permitting** c.e. Turing degrees:

- decomposition of c.e. T-degrees into definable filter and definable ideal

- characterisation of structural properties:

Theorem (Ambos-Spies, Jockusch, Shore, Soare 1984)

For a c.e. degree $\mathbf{a}$*, TFAE:*

- $\mathbf{a}$ *is promptly permitting;*
- $\mathbf{a}$ *is non-cappable:* $\nexists \mathbf{b} > \mathbf{0}$ *s.t.* $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$*;*
- $\mathbf{a}$ *is low cuppable:* $\exists \mathbf{b}$, $\mathbf{b}' = \mathbf{0}'$, $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$*.*

Variants, strengthenings of these notions:

- prompt versions of other forms of permitting?
- strengthenings or weakenings of capping or cupping properties?

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There is a universal c.e. operator U with weight $U^X < 1$ such that

$$A \text{ is low-for-random } \Leftrightarrow \exists \text{ c.e. } V \text{ s.t. } U^A \subseteq V \text{ and weight } V < \infty.$$

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- threaten to make $U^A \subseteq V$
- guarantees enough A-changes to make weight $V = \infty$.

Prompt non-low-for-randomness

Analogously with prompt permitting, we can define prompt non-low-for-random permitting:

A is **promptly non-low-for-random** (PNLFR) if $\exists$ computable p s.t. if $U^A \subseteq V$ then the set of σ such that

$$\sigma \in V[\text{at } s], \qquad \sigma \in U^A[s] \text{ with use } u, \qquad A[s] \upharpoonright u \neq A[p(s)] \upharpoonright u$$

has infinite weight.

- PNLFR's exist, including low, Turing-complete

- PNLFR implies promptly permitting

- proper subclass of promptly permitting non-low-for-random c.e. sets

- closed upwards under $\leq_T$ but unknown if it forms a filter

Strong promptness

Another strengthening of prompt permitting by Diamondstone, Ng:

A is **strongly prompt** if there are computable p, ω-c.e. g s.t.

$$|W_e| \geq g(e) \;\Rightarrow\; \exists x, s : x \in W[\text{at } s] \text{ and } A[s] \upharpoonright x \neq A[p(s)] \upharpoonright x.$$

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What is the connection between strong promptness and prompt non-low-for-randomness?

$\exists$ Turing-complete not strongly prompt set [Diamondstone, Ng]
$\to$ promptly non-low-for-random $\not\Rightarrow$ strongly prompt

$\exists$ low-for-random strongly prompt set [Diamondstone, Ng]
$\to$ strongly prompt $\not\Rightarrow$ promptly non-low-for-random

$\exists$ non-low-for-random, strongly prompt,
not promptly non-low-for-random set

Capping properties

Recall: prompt permitting $\Leftrightarrow$ non-cappable

Are there analogous results for PNLFR, strong promptness?

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- Converse unknown

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Strongly prompt and superlow cupping

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$$\text{strongly prompt} \;\Leftrightarrow\; \text{superlow cuppable via reduction with } \omega\text{-c.e. use}$$

However,

$$\text{strongly prompt} \;\not\Rightarrow\; \text{LfR cuppable via reduction with } \omega\text{-c.e. use}$$

Further work

- promptly non-low-for-random $\Rightarrow$ not cappable to LfR?
- promptly non-low-for-random $\Rightarrow$ LfR cuppable?
- other notions: cuppable to LR-complete/AED?
- strong version of prompt non-low-for-randomness?